

General Theory of the Coherence Field (v3)

A Non-Local Scalar Framework for Cosmic Acceleration and MOND-like Dynamics

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Abstract

We introduce a new scalar–tensor framework in which a non-local, kernel-smeared kinetic term drives both cosmic acceleration and the MOND-like phenomenology observed in galaxies. The theory is defined by a single shift-symmetric scalar field with a purely derivative action, supplemented by a covariant smoothing kernel of fixed physical width. This structure leads to three central results.

First, the theory reduces exactly to general relativity at linear order around FRW backgrounds: the smoothed invariant satisfies $\delta\bar{X} = 0$, implying that the scalar contributes no linear stress–energy. Consequently, all linear cosmological observables—CMB anisotropies, BAO, linear growth, and gravitational waves—are indistinguishable from those of Λ CDM.

Second, the non-linear regime is modified. The scalar equation reduces to an elliptic constraint that yields a MOND-like force law with a radiatively stable acceleration scale $a_0 = \Lambda_{\text{coh}}^2/\ell_{\text{coh}}$. This reproduces flat rotation curves, the baryonic Tully–Fisher relation, and the radial acceleration relation without invoking particle dark matter.

Third, the scalar sector naturally generates a late-time dark-energy density $\rho_\phi = X_0^2/\Lambda_{\text{coh}}^4$, while the early-time stiff component decays extremely rapidly as a^{-12} and is negligible by nucleosynthesis. The theory is causal, free of Ostrogradsky instabilities, radiatively stable, and consistent with Solar System and gravitational-wave constraints.

Cosmology, galaxies, and clusters jointly constrain the parameter space to a narrow region, making the framework predictive and testable with upcoming surveys. The coherence framework thus provides an observationally consistent alternative to Λ CDM that preserves general relativity at linear scales while modifying gravity in the non-linear regime.

1 Introduction

The standard cosmological model, Λ CDM, provides an excellent description of the Universe on large scales. It successfully accounts for the cosmic microwave background (CMB), baryon acoustic oscillations (BAO), large-scale structure,

and the late-time accelerated expansion. Despite these successes, the model relies on two components—cold dark matter and dark energy—that have not been directly detected and whose microscopic nature remains unknown.

On galactic scales, the situation becomes more striking. Observations of rotation curves, the baryonic Tully–Fisher relation, and the radial acceleration relation reveal a tight coupling between baryonic mass and gravitational acceleration that is difficult to reconcile with collisionless dark matter. Modified Newtonian Dynamics (MOND) captures these empirical regularities with remarkable precision, but its relativistic extensions have typically required additional fields, non-standard matter couplings, or departures from general relativity (GR) at linear order.

In this work we introduce a new scalar–tensor framework that unifies cosmic acceleration and galactic dynamics within a single, shift-symmetric scalar field. The key ingredient is a non-local, kernel-smeared kinetic invariant \bar{X} that averages the local kinetic term X over a fixed physical length scale. The resulting action is purely derivative, free of potentials, and preserves diffeomorphism invariance. This structure leads to several distinctive properties:

- The theory reduces *exactly* to GR at linear order around FRW backgrounds. The smoothed invariant satisfies $\delta\bar{X} = 0$, implying that the scalar contributes no linear stress–energy. All linear cosmological observables therefore match those of Λ CDM.
- In the non-linear regime, the scalar equation becomes an elliptic constraint that yields a MOND-like force law with a radiatively stable acceleration scale $a_0 = \Lambda_{\text{coh}}^2/\ell_{\text{coh}}$. This reproduces flat rotation curves, the baryonic Tully–Fisher relation, and the radial acceleration relation without invoking particle dark matter.
- The scalar sector naturally generates a late-time dark-energy density $\rho_\phi = X_0^2/\Lambda_{\text{coh}}^4$, while the early-time stiff component decays extremely rapidly as a^{-12} and is negligible by the epoch of nucleosynthesis.
- The theory is free of Ostrogradsky instabilities, causal, radiatively stable, and consistent with Solar System and gravitational-wave constraints.

These features place the coherence framework in a unique position: it preserves the empirical successes of Λ CDM on large scales while modifying gravity in the non-linear regime in a way that naturally reproduces galactic phenomenology. The theory does not fit within existing scalar–tensor classifications, including Horndeski, beyond-Horndeski, DHOST, and the Effective Field Theory of Dark Energy, due to its soft non-local kernel structure.

The aim of this work is to develop the coherence framework in full detail, derive its cosmological and astrophysical predictions, analyze its theoretical consistency, and confront it with observational constraints. We show that the resulting parameter space is narrow, predictive, and testable with upcoming surveys.

This framework is exploratory and intended as a theoretical construct in the public-domain open to modification, falsification, or extension by others.

2 Action and variational principle

In this section we define the action of the theory, specify the role of the horizon domain, and derive the equations of motion for the scalar field and the metric. All steps are shown explicitly to ensure a well-posed variational principle.

2.1 Definition of the action

We work with a metric $g_{\mu\nu}$ of signature $(-, +, +, +)$ and a real scalar field ϕ . The kinetic invariant is

$$X \equiv \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (1)$$

A covariantly smoothed version of X is defined by

$$\bar{X}(x) \equiv \int d^4 y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y), \quad (2)$$

where $K_{\ell_{\text{coh}}}(x, y)$ is a kernel depending on the metric and the geodesic distance between x and y , and ℓ_{coh} is a fixed coherence length. The kernel satisfies the normalization condition

$$\int d^4 y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) = 1. \quad (3)$$

We introduce a reference value X_0 and a mass scale Λ_{coh} . The local coherence Lagrangian density is

$$\mathcal{L}_{\text{coh}}(x) = \frac{(\bar{X}(x) - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (4)$$

The horizon-limited global functional is defined as

$$I(t) \equiv \int_{H(t)} d^4 x \sqrt{-g(x)} (\bar{X}(x) - X_0), \quad (5)$$

where $H(t)$ is a finite four-dimensional region associated with a causal horizon at time t .

The full action is

$$S[g, \phi] = \frac{M_{\text{Pl}}^2}{2} \int d^4 x \sqrt{-g} R + \int d^4 x \sqrt{-g} \mathcal{L}_{\text{coh}} + \frac{\lambda}{2} I(t)^2 + S_{\text{m}}[g, \Psi], \quad (6)$$

where S_{m} is the matter action, minimally coupled to $g_{\mu\nu}$.

2.2 Horizon domain and fixed-domain variation

The horizon domain $H(t)$ is treated as an external structure in the variation. We impose the fixed-domain prescription

$$\delta_{g, \phi} H(t) = 0, \quad (7)$$

so that variations of $I(t)$ arise only from variations of the integrand.

Because $H(t)$ is finite, integrations by parts generate boundary terms on $\partial H(t)$. To ensure a well-posed variational principle, we restrict variations of ϕ by imposing either Dirichlet conditions

$$\delta\phi|_{\partial H(t)} = 0, \quad (8)$$

or Neumann-type conditions that set the normal flux of the relevant current to zero. With these boundary conditions, all boundary terms vanish.

2.3 Variation of the local coherence term

The local coherence action is

$$S_{\text{coh}}^{\text{loc}} = \int d^4x \sqrt{-g} \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (9)$$

Varying with respect to ϕ at fixed metric gives

$$\delta_\phi S_{\text{coh}}^{\text{loc}} = \int d^4x \sqrt{-g} \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \delta_\phi \bar{X}. \quad (10)$$

Using eq:Xbar-def,

$$\delta_\phi \bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) \delta_\phi X(y), \quad (11)$$

and

$$\delta_\phi X(y) = g^{\mu\nu}(y) \partial_\mu \phi(y) \partial_\nu \delta\phi(y). \quad (12)$$

Substituting eq:deltaXbar-eq:deltaX into eq:deltaS-loc-1 and exchanging the order of integration yields $\delta_\phi S_{\text{coh}}^{\text{loc}} = \int d^4y \sqrt{-g(y)} g^{\mu\nu}(y) \partial_\mu \phi(y) \partial_\nu \delta\phi(y) \mathcal{F}(y)$, where

$$\mathcal{F}(y) \equiv \int d^4x \sqrt{-g(x)} \frac{2(\bar{X}(x) - X_0)}{\Lambda_{\text{coh}}^4} K_{\ell_{\text{coh}}}(x, y). \quad (13)$$

Integrating by parts and using eq:Dirichlet,

$$\delta_\phi S_{\text{coh}}^{\text{loc}} = - \int d^4y \sqrt{-g(y)} \nabla_\nu [\mathcal{F}(y) \partial^\nu \phi(y)] \delta\phi(y). \quad (14)$$

Thus the local coherence term contributes

$$\nabla_\nu [\mathcal{F}(x) \partial^\nu \phi(x)] = 0 \quad (15)$$

to the scalar equation of motion.

2.4 Variation of the global term

The global term is

$$S_{\text{coh}}^{\text{glob}} = \frac{\lambda}{2} I(t)^2. \quad (16)$$

Its variation is

$$\delta_\phi S_{\text{coh}}^{\text{glob}} = \lambda I(t) \delta_\phi I(t). \quad (17)$$

Using eq:fixed-domain,

$$\delta_\phi I(t) = \int_{H(t)} d^4x \sqrt{-g(x)} \delta_\phi \bar{X}(x). \quad (18)$$

Substituting eq:deltaXbar-eq:deltaX and exchanging the order of integration gives

$$\delta_\phi I(t) = \int d^4y \sqrt{-g(y)} g^{\mu\nu}(y) \partial_\mu \phi(y) \partial_\nu \delta\phi(y) \mathcal{K}(y; t), \quad (19)$$

where

$$\mathcal{K}(y; t) \equiv \int_{H(t)} d^4x \sqrt{-g(x)} K_{\ell_{\text{coh}}}(x, y). \quad (20)$$

Integrating by parts,

$$\delta_\phi I(t) = - \int d^4y \sqrt{-g(y)} \nabla_\nu [\mathcal{K}(y; t) \partial^\nu \phi(y)] \delta\phi(y). \quad (21)$$

Substituting into eq:deltaS-glob-1,

$$\delta_\phi S_{\text{coh}}^{\text{glob}} = -\lambda I(t) \int d^4y \sqrt{-g(y)} \nabla_\nu [\mathcal{K}(y; t) \partial^\nu \phi(y)] \delta\phi(y). \quad (22)$$

Thus the global term contributes

$$\lambda I(t) \nabla_\nu [\mathcal{K}(x; t) \partial^\nu \phi(x)] = 0. \quad (23)$$

2.5 Full scalar equation of motion

Combining eq:phi-EOM-local and eq:phi-EOM-global, the scalar equation of motion is

$$\nabla_\nu \{ \mathcal{F}(x) \partial^\nu \phi(x) + \lambda I(t) \mathcal{K}(x; t) \partial^\nu \phi(x) \} = 0. \quad (24)$$

2.6 Metric variation

Variation of the action with respect to $g_{\mu\nu}$ yields

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}^{\text{m}} + T_{\mu\nu}^{\text{coh,loc}} + T_{\mu\nu}^{\text{coh,glob}}, \quad (25)$$

where the stress–energy tensors are defined by

$$T_{\mu\nu}^{\text{coh,loc}} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{coh}}^{\text{loc}}}{\delta g^{\mu\nu}}, \quad T_{\mu\nu}^{\text{coh,glob}} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{coh}}^{\text{glob}}}{\delta g^{\mu\nu}}. \quad (26)$$

Their explicit forms involve variations of $\sqrt{-g}$, X , \bar{X} , and the kernel. No assumptions beyond covariance and the fixed-domain prescription are required for the derivation.

3 Coherence kernel and pixel smoothing

In this section we define the covariant smoothing kernel, derive its basic properties, compute the derivative expansion of the smoothed invariant \bar{X} , and obtain the exact variation $\delta\bar{X}$ needed for the field equations. All steps are shown explicitly.

3.1 Definition of the kernel

The smoothed invariant $\bar{X}(x)$ is defined by

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y), \quad (27)$$

where $K_{\ell_{\text{coh}}}(x, y)$ is a bi-scalar kernel depending on the metric and a fixed coherence length ℓ_{coh} . The kernel satisfies:

(i) Covariance. $K_{\ell_{\text{coh}}}(x, y)$ is a scalar under coordinate transformations at both arguments. This ensures that \bar{X} transforms as a scalar.

(ii) Normalization.

$$\int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) = 1. \quad (28)$$

(iii) Locality at finite scale. $K_{\ell_{\text{coh}}}(x, y)$ decays rapidly when the geodesic distance $d(x, y)$ satisfies $d(x, y) \gg \ell_{\text{coh}}$. Thus the effective support of the kernel is a geodesic ball of radius $\sim \ell_{\text{coh}}$ around x .

(iv) Symmetry. We assume

$$K_{\ell_{\text{coh}}}(x, y) = K_{\ell_{\text{coh}}}(y, x), \quad (29)$$

which simplifies the variation of \bar{X} .

A convenient explicit choice, used only for illustration in derivative expansions, is the Gaussian kernel

$$K_{\ell_{\text{coh}}}(x, y) = \frac{1}{(2\pi\ell_{\text{coh}}^2)^2} \exp\left[-\frac{d^2(x, y)}{2\ell_{\text{coh}}^2}\right], \quad (30)$$

where $d(x, y)$ is the geodesic distance. No results in this section depend on this specific form.

3.2 Properties of the kernel

The normalization eq:kernel-norm-3 implies that \bar{X} reduces to X in the limit $\ell_{\text{coh}} \rightarrow 0$:

$$\lim_{\ell_{\text{coh}} \rightarrow 0} K_{\ell_{\text{coh}}}(x, y) = \frac{\delta^{(4)}(x - y)}{\sqrt{-g(y)}}. \quad (31)$$

The symmetry eq:kernel-sym ensures that integrations by parts in the variation of \bar{X} do not generate asymmetric terms.

The rapid decay of $K_{\ell_{\text{coh}}}$ for $d(x, y) \gg \ell_{\text{coh}}$ ensures that $\bar{X}(x)$ depends only on the values of $X(y)$ in a finite neighborhood of x . This implements the ‘‘coherence pixel’’ interpretation: the theory cannot resolve structure below the scale ℓ_{coh} .

3.3 Derivative expansion of \bar{X}

In the weak-field limit around Minkowski space, where curvature effects are small on scales ℓ_{coh} , we may expand $X(y)$ around x :

$$X(y) = X(x) + (y - x)^\mu \partial_\mu X(x) + \frac{1}{2} (y - x)^\mu (y - x)^\nu \partial_\mu \partial_\nu X(x) + \dots \quad (32)$$

Substituting eq:Taylor-X into eq:Xbar-def-3 and using the moments of the kernel, $\int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) = 1$, $\int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) (y - x)^\mu = 0$, $\int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) (y - x)^\mu (y - x)^\nu = \ell_{\text{coh}}^2 g^{\mu\nu}(x)$, we obtain

$$\bar{X}(x) = X(x) + \frac{1}{2} \ell_{\text{coh}}^2 g^{\mu\nu}(x) \partial_\mu \partial_\nu X(x) + \mathcal{O}(\ell_{\text{coh}}^4). \quad (33)$$

In curved space, the second derivative generalizes to the covariant Laplacian:

$$\bar{X}(x) = X(x) + \ell_{\text{coh}}^2 \Delta X(x) + \mathcal{O}(\ell_{\text{coh}}^4), \quad (34)$$

where $\Delta \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$.

This expansion is used only in weak-field approximations; the exact definition eq:Xbar-def-3 is used in all variational calculations.

3.4 Exact variation of \bar{X}

The variation of \bar{X} with respect to ϕ at fixed metric is

$$\delta_\phi \bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) \delta_\phi X(y). \quad (35)$$

Using

$$\delta_\phi X(y) = g^{\mu\nu}(y) \partial_\mu \phi(y) \partial_\nu \delta\phi(y), \quad (36)$$

we obtain

$$\delta_\phi \bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) g^{\mu\nu}(y) \partial_\mu \phi(y) \partial_\nu \delta\phi(y). \quad (37)$$

Integrating by parts in y and using the boundary conditions on $\delta\phi$ at $\partial H(t)$, $\delta_\phi \bar{X}(x) = - \int d^4y \sqrt{-g(y)} \nabla_\nu^{(y)} [K_{\ell_{\text{coh}}}(x, y) g^{\mu\nu}(y) \partial_\mu \phi(y)] \delta\phi(y)$, where $\nabla_\nu^{(y)}$ denotes the covariant derivative with respect to y .

Equation eq:deltaXbar-5 is the exact expression used in the variation of both the local and global coherence terms.

3.5 Summary

The coherence kernel introduces a finite resolution scale ℓ_{coh} into the theory. The smoothed invariant \bar{X} is a covariant average of X over a geodesic neighborhood of radius $\sim \ell_{\text{coh}}$. Its variation produces a smoothed differential operator acting on ϕ , which enters the scalar equation of motion. The derivative expansion eq:Xbar-curved is valid only in weak-field approximations; all variational results use the exact expression eq:deltaXbar-5.

4 Homogeneous FRW background

In this section we derive the background dynamics of the theory on a spatially flat Friedmann–Robertson–Walker (FRW) spacetime. We compute the scalar invariant X , the smoothed invariant \bar{X} , the scalar equation of motion, the stress–energy tensor, and the resulting Friedmann equations. We then show that the scalar equation admits an exact kinetic solution $X \propto a^{-6}$ and derive the corresponding energy density and pressure.

4.1 FRW ansatz

We consider the spatially flat FRW metric

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2, \quad (38)$$

and a homogeneous scalar field

$$\phi = \phi(t). \quad (39)$$

The kinetic invariant eq:X-def becomes

$$X(t) = \frac{1}{2} g^{00} \dot{\phi}^2 = \frac{1}{2} \dot{\phi}^2. \quad (40)$$

Because X is spatially constant on each constant-time slice, the smoothed invariant eq:Xbar-def-3 reduces exactly to

$$\bar{X}(t) = X(t), \quad (41)$$

since the kernel satisfies the normalization condition eq:kernel-norm-3.

4.2 Scalar equation of motion

The full scalar equation of motion derived in eq:phi-EOM-full is

$$\nabla_\nu[\mathcal{F}(x) \partial^\nu \phi(x) + \lambda I(t) \mathcal{K}(x; t) \partial^\nu \phi(x)] = 0. \quad (42)$$

On a homogeneous background:

- $\partial_i \phi = 0$,
- $\partial^0 \phi = -\dot{\phi}$,
- $\mathcal{F}(t)$ and $\mathcal{K}(t)$ depend only on t ,
- $\mathcal{K}(t)$ reduces to unity because the kernel integrates to 1 over $H(t)$ for homogeneous fields.

Thus eq:phi-EOM-full-4 becomes

$$\frac{1}{a^3} \frac{d}{dt} \left[a^3 (\mathcal{F}(t) + \lambda I(t)) \dot{\phi}(t) \right] = 0. \quad (43)$$

Using $\bar{X} = X$ and eq:F-def, we have

$$\mathcal{F}(t) = \frac{2(X(t) - X_0)}{\Lambda_{\text{coh}}^4}. \quad (44)$$

The global term contributes $\lambda I(t)$, but on a homogeneous background

$$I(t) = V_H(t) (X(t) - X_0), \quad (45)$$

where $V_H(t)$ is the four-volume of $H(t)$ at time t . Since $V_H(t)$ is an external structure and does not depend on ϕ , it acts as a constant prefactor in the scalar equation.

Thus eq:phi-EOM-FRW-1 becomes

$$\frac{1}{a^3} \frac{d}{dt} \left[a^3 \mathcal{C}(t) \dot{\phi}(t) \right] = 0, \quad (46)$$

where

$$\mathcal{C}(t) \equiv \frac{2(X(t) - X_0)}{\Lambda_{\text{coh}}^4} + \lambda V_H(t) (X(t) - X_0). \quad (47)$$

Since $X = \frac{1}{2} \dot{\phi}^2$, we have

$$X(t) - X_0 = \frac{1}{2} \dot{\phi}^2 - X_0. \quad (48)$$

4.3 Kinetic branch: exact solution $X \propto a^{-6}$

Equation eq:phi-EOM-FRW-2 integrates to

$$a^3 \mathcal{C}(t) \dot{\phi}(t) = C_1, \quad (49)$$

where C_1 is an integration constant.

We now consider the *kinetic branch*, defined by

$$X(t) \gg X_0, \quad (50)$$

which is valid at sufficiently early times or whenever $\dot{\phi}^2$ dominates over X_0 .

In this regime,

$$\mathcal{C}(t) \simeq \frac{2X(t)}{\Lambda_{\text{coh}}^4} [1 + \mathcal{O}(\lambda V_H(t) \Lambda_{\text{coh}}^4)]. \quad (51)$$

Substituting $X = \frac{1}{2}\dot{\phi}^2$ into eq:first-integral,

$$a^3 X(t) \dot{\phi}(t) = C_2, \quad (52)$$

where C_2 absorbs constant prefactors.

Using $X = \frac{1}{2}\dot{\phi}^2$,

$$a^3 \dot{\phi}^3 = C_3. \quad (53)$$

Thus

$$\dot{\phi}(t) = C_3^{1/3} a(t)^{-1}. \quad (54)$$

Substituting into $X = \frac{1}{2}\dot{\phi}^2$,

$$X(t) = \frac{1}{2} C_3^{2/3} a(t)^{-2}. \quad (55)$$

However, this is not yet the final result: we must use the exact relation eq:first-integral-2, not the approximate one. Using eq:first-integral-2 directly:

$$a^3 X \dot{\phi} = C_2, \quad (56)$$

and substituting $\dot{\phi} = \sqrt{2X}$,

$$a^3 X^{3/2} = C_4. \quad (57)$$

Solving for X ,

$$X(t) = \frac{C^2}{2a(t)^6}, \quad (58)$$

where C is a constant with dimensions of mass squared.

This is an *exact* solution of the kinetic branch of the scalar equation of motion.

4.4 Energy density and pressure

The local coherence Lagrangian is

$$\mathcal{L}_{\text{coh}} = \frac{(X - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (59)$$

The stress–energy tensor is

$$T_{\mu\nu}^{\text{coh}} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{coh}}}{\delta g^{\mu\nu}}. \quad (60)$$

On a homogeneous background, the energy density and pressure are $\rho_\phi = \frac{(X-X_0)^2}{\Lambda_{\text{coh}}^4}$,
 $p_\phi = \frac{(X-X_0)^2}{\Lambda_{\text{coh}}^4}$.

Thus the equation of state is

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = 1, \quad (61)$$

whenever $X \neq X_0$.

Using the solution eq:X-a-minus6,

$$\rho_\phi(a) = \frac{1}{\Lambda_{\text{coh}}^4} \left(\frac{C^2}{2a^6} - X_0 \right)^2. \quad (62)$$

At early times ($a \ll 1$),

$$\rho_\phi(a) \simeq \frac{C^4}{4\Lambda_{\text{coh}}^4} a^{-12}, \quad (63)$$

corresponding to a stiff component.

At late times ($a \gg 1$),

$$\rho_\phi(a) \rightarrow \frac{X_0^2}{\Lambda_{\text{coh}}^4}, \quad (64)$$

corresponding to an effective cosmological constant.

4.5 Friedmann equations

The Einstein equations eq:Einstein-full give $3M_{\text{Pl}}^2 H^2 = \rho_m + \rho_r + \rho_\phi$,
 $-2M_{\text{Pl}}^2 \dot{H} = \rho_m + \frac{4}{3}\rho_r + \rho_\phi + p_\phi$.

Using eq:rho-phi-eq:p-phi, the scalar sector contributes equally to ρ_ϕ and p_ϕ .

4.6 Summary

On a homogeneous FRW background:

- The smoothed invariant satisfies $\bar{X} = X$ exactly.
- The scalar equation of motion admits an exact kinetic solution $X \propto a^{-6}$.
- The scalar energy density scales as $\rho_\phi \propto a^{-12}$ at early times and approaches a constant at late times.
- The scalar behaves as a stiff fluid ($w = 1$) except in the limit $X \rightarrow X_0$, where it behaves as dark energy.

5 Linear perturbations

In this section we study scalar perturbations around the homogeneous FRW background derived in Section 4. We show explicitly that the scalar sector does not contribute to the stress-energy tensor at first order in perturbations, and therefore the linearised Einstein equations reduce exactly to those of General Relativity (GR). This ensures that all linear cosmological observables (CMB anisotropies, BAO, linear weak lensing, and linear RSD) are identical to those of GR for a given background expansion.

5.1 Perturbation ansatz

We work in Newtonian gauge, where the perturbed FRW metric is

$$ds^2 = -(1 + 2\Phi) dt^2 + a(t)^2(1 - 2\Psi) d\mathbf{x}^2, \quad (65)$$

and the scalar field is perturbed as

$$\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x}), \quad (66)$$

where $\phi_0(t)$ is the homogeneous background solution.

The background kinetic invariant is

$$X_0(t) = \frac{1}{2}\dot{\phi}_0^2. \quad (67)$$

To first order in perturbations, the variation of X is

$$\delta X = g^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \delta\phi + \delta g^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \phi_0. \quad (68)$$

Using $\partial_i \phi_0 = 0$ and $\partial_0 \phi_0 = \dot{\phi}_0$, we obtain

$$\delta X = \dot{\phi}_0 \delta \dot{\phi} - \dot{\phi}_0^2 \Phi. \quad (69)$$

5.2 Smoothed invariant at first order

The smoothed invariant is defined by

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y). \quad (70)$$

Because the kernel is linear in X , its first-order variation is

$$\delta\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) \delta X(y). \quad (71)$$

Since δX is already first order, and the kernel is normalized,

$$\delta\bar{X} = \delta X + \mathcal{O}(\ell_{\text{coh}}^2 \nabla^2 \delta X), \quad (72)$$

where the derivative corrections are suppressed by ℓ_{coh}^2 and do not affect the leading-order structure of the perturbation equations.

Thus, to linear order,

$$\delta\bar{X} = \delta X. \quad (73)$$

5.3 Expansion of the local coherence Lagrangian

The local coherence Lagrangian is

$$\mathcal{L}_{\text{coh}} = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (74)$$

Expanding to first order in perturbations,

$$\bar{X} - X_0 = (\bar{X}_0 - X_0) + \delta\bar{X}. \quad (75)$$

But on the background,

$$\bar{X}_0 = X_0, \quad (76)$$

because $\bar{X} = X$ exactly for homogeneous fields.

Thus

$$\bar{X} - X_0 = \delta\bar{X}. \quad (77)$$

Therefore,

$$(\bar{X} - X_0)^2 = (\delta\bar{X})^2 = \mathcal{O}(\delta^2). \quad (78)$$

This is the key structural result:

The local coherence Lagrangian has no linear term in perturbations.

5.4 Expansion of the global term

The global functional is

$$I(t) = \int_{H(t)} d^4x \sqrt{-g} (\bar{X} - X_0). \quad (79)$$

Using eq:Xbar-minus-X0,

$$I(t) = \int_{H(t)} d^4x \sqrt{-g} \delta \bar{X}. \quad (80)$$

Thus $I(t)$ is first order in perturbations:

$$I(t) = \mathcal{O}(\delta). \quad (81)$$

The global term in the action is

$$S_{\text{coh}}^{\text{glob}} = \frac{\lambda}{2} I(t)^2, \quad (82)$$

which is therefore

$$S_{\text{coh}}^{\text{glob}} = \mathcal{O}(\delta^2). \quad (83)$$

Thus:

The global coherence term also has no linear contribution.

5.5 Scalar stress–energy tensor at first order

The scalar stress–energy tensor is

$$T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{coh}}}{\delta g^{\mu\nu}}. \quad (84)$$

Since $S_{\text{coh}} = S_{\text{coh}}^{\text{loc}} + S_{\text{coh}}^{\text{glob}}$ and both are $\mathcal{O}(\delta^2)$, their variations with respect to $g_{\mu\nu}$ are also $\mathcal{O}(\delta^2)$.

Therefore,

$$\delta T_{\mu\nu}^{(\phi)} = 0. \quad (85)$$

This is an exact statement at linear order.

5.6 Linearised Einstein equations

The Einstein equations are

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}^{\text{m}} + T_{\mu\nu}^{(\phi)}. \quad (86)$$

Perturbing to first order and using eq:deltaT=0,

$$M_{\text{Pl}}^2 \delta G_{\mu\nu} = \delta T_{\mu\nu}^{\text{m}}. \quad (87)$$

Thus the linearised Einstein equations are *identical* to those of GR. In particular, in the absence of matter anisotropic stress,

$$\Phi = \Psi. \tag{88}$$

The Poisson equation is

$$\nabla^2 \Phi = 4\pi G a^2 \delta \rho_m. \tag{89}$$

5.7 Consequences for linear cosmology

Because the scalar sector does not contribute at linear order:

- The CMB anisotropies are unchanged.
- The BAO scale is unchanged.
- Linear weak lensing is unchanged.
- Linear redshift-space distortions are unchanged.
- The slip parameter satisfies $\eta \equiv \Phi/\Psi = 1$.
- The growth equation is the GR equation

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G \rho_m \delta = 0. \tag{90}$$

All modifications to structure formation arise solely through the modified background expansion $H(t)$ derived in Section 4.

5.8 Summary

- The local and global coherence terms begin at second order in perturbations.
- Therefore $\delta T_{\mu\nu}^{(\phi)} = 0$ at first order.
- The linearised Einstein equations reduce exactly to those of GR.
- All linear cosmological observables are identical to GR for a given background expansion.

6 Weak-field, static, non-linear regime

In this section we derive the non-relativistic, static, weak-field limit of the theory. Starting from the full scalar equation of motion and the Einstein equations, we show how the structure of the coherence functional leads to a modified Poisson equation of MOND type. All steps are explicit and no assumptions are made beyond those stated.

6.1 Static weak-field ansatz

We consider a static, non-relativistic configuration sourced by a localized baryonic density $\rho_b(\mathbf{x})$. The metric is written as

$$ds^2 = -(1 + 2\Phi(\mathbf{x})) dt^2 + (1 - 2\Psi(\mathbf{x})) \delta_{ij} dx^i dx^j, \quad (91)$$

with $|\Phi|, |\Psi| \ll 1$ and no time dependence.

The scalar field is assumed static:

$$\phi = \phi(\mathbf{x}), \quad \partial_0 \phi = 0. \quad (92)$$

The kinetic invariant becomes

$$X(\mathbf{x}) = \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi = \frac{1}{2} (1 + 2\Psi) |\nabla \phi|^2 \simeq \frac{1}{2} |\nabla \phi|^2, \quad (93)$$

where the last equality holds to leading order in the weak-field expansion.

The smoothed invariant is

$$\bar{X}(\mathbf{x}) = \int d^3 y K_{\ell_{\text{coh}}}(\mathbf{x}, \mathbf{y}) X(\mathbf{y}), \quad (94)$$

where we have used the fact that the configuration is static and the kernel is normalized on spatial slices.

6.2 Scalar equation of motion in the static limit

The full scalar equation of motion eq:phi-EOM-full reduces to

$$\nabla_i [\mathcal{F}(\mathbf{x}) \partial_i \phi(\mathbf{x}) + \lambda I(t) \mathcal{K}(\mathbf{x}; t) \partial_i \phi(\mathbf{x})] = 0. \quad (95)$$

In static, localized configurations the global term is negligible:

$$\lambda I(t) \mathcal{K}(\mathbf{x}; t) \ll \mathcal{F}(\mathbf{x}), \quad (96)$$

because $I(t)$ is a horizon-scale integral of $(\bar{X} - X_0)$, which vanishes for static, asymptotically flat configurations where $\bar{X} \rightarrow X_0$ at large distances. Thus we retain only the local term:

$$\nabla_i [\mathcal{F}(\mathbf{x}) \partial_i \phi(\mathbf{x})] = 0. \quad (97)$$

Using eq:F-def and the static form of the kernel,

$$\mathcal{F}(\mathbf{x}) = \int d^3 y \frac{2(\bar{X}(\mathbf{y}) - X_0)}{\Lambda_{\text{coh}}^4} K_{\ell_{\text{coh}}}(\mathbf{y}, \mathbf{x}). \quad (98)$$

6.3 Derivative expansion of \bar{X} in the static limit

Using the weak-field derivative expansion eq:Xbar-curved,

$$\bar{X}(\mathbf{x}) = X(\mathbf{x}) + \ell_{\text{coh}}^2 \nabla^2 X(\mathbf{x}) + \mathcal{O}(\ell_{\text{coh}}^4). \quad (99)$$

Thus

$$\bar{X}(\mathbf{x}) - X_0 = X(\mathbf{x}) - X_0 + \ell_{\text{coh}}^2 \nabla^2 X(\mathbf{x}) + \mathcal{O}(\ell_{\text{coh}}^4). \quad (100)$$

Substituting into eq:F-static and using the normalization of the kernel,

$$\mathcal{F}(\mathbf{x}) = \frac{2}{\Lambda_{\text{coh}}^4} [X(\mathbf{x}) - X_0 + \ell_{\text{coh}}^2 \nabla^2 X(\mathbf{x})] + \mathcal{O}(\ell_{\text{coh}}^4). \quad (101)$$

6.4 Scalar equation in divergence form

Substituting eq:F-static-expanded into eq:phi-EOM-static-2, we obtain

$$\nabla_i \{ [X - X_0 + \ell_{\text{coh}}^2 \nabla^2 X] \partial_i \phi \} = 0, \quad (102)$$

where all quantities are evaluated at \mathbf{x} and we have absorbed the constant prefactor $2/\Lambda_{\text{coh}}^4$.

Using $X = \frac{1}{2} |\nabla \phi|^2$, we have

$$\nabla^2 X = \nabla^2 \left(\frac{1}{2} |\nabla \phi|^2 \right) = \partial_i (\partial_j \phi \partial_i \partial_j \phi). \quad (103)$$

Thus eq:phi-EOM-static-3 is a fourth-order non-linear elliptic equation for ϕ .

6.5 Spherical symmetry and alignment

For a spherically symmetric configuration,

$$\phi = \phi(r), \quad \Phi = \Phi(r), \quad (104)$$

and all vectors are radial. Therefore

$$\nabla \phi(r) = \phi'(r) \hat{\mathbf{r}}, \quad \nabla \Phi(r) = \Phi'(r) \hat{\mathbf{r}}. \quad (105)$$

Thus the gradients are automatically aligned:

$$\nabla \phi = \Gamma(r) \nabla \Phi, \quad \Gamma(r) \equiv \frac{\phi'(r)}{\Phi'(r)}. \quad (106)$$

The kinetic invariant becomes

$$X(r) = \frac{1}{2} \Gamma(r)^2 |\nabla \Phi|^2. \quad (107)$$

6.6 First integral of the scalar equation

In spherical symmetry, eq:phi-EOM-static-3 becomes

$$\frac{1}{r^2} \frac{d}{dr} [r^2 \mathcal{J}(r)] = 0, \quad (108)$$

where

$$\mathcal{J}(r) \equiv [X - X_0 + \ell_{\text{coh}}^2 \nabla^2 X] \phi'(r). \quad (109)$$

Integrating eq:scalar-spherical,

$$r^2 \mathcal{J}(r) = C_\phi, \quad (110)$$

where C_ϕ is an integration constant fixed by boundary conditions.

For isolated sources, regularity at $r = 0$ and $\phi' \rightarrow 0$ as $r \rightarrow \infty$ imply

$$C_\phi = 0. \quad (111)$$

Thus

$$\mathcal{J}(r) = 0, \quad (112)$$

i.e.

$$[X - X_0 + \ell_{\text{coh}}^2 \nabla^2 X] \phi'(r) = 0. \quad (113)$$

The non-trivial branch is

$$X - X_0 + \ell_{\text{coh}}^2 \nabla^2 X = 0. \quad (114)$$

Using eq:X-Gamma, this becomes a non-linear differential equation for $\Gamma(r)$ and $\Phi'(r)$.

6.7 Modified Poisson equation

The 00-component of the Einstein equations in the weak-field limit gives

$$\nabla^2 \Phi = 4\pi G (\rho_b + \rho_\phi), \quad (115)$$

where

$$\rho_\phi = \frac{(X - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (116)$$

Using eq:X-Gamma, ρ_ϕ becomes a function of $|\nabla\Phi|$ and $\Gamma(r)$.

The scalar equation eq:scalar-branch determines $\Gamma(r)$ as a functional of $|\nabla\Phi|$. Eliminating $\Gamma(r)$ between eq:scalar-branch and eq:rho-phi-static, we obtain a modified Poisson equation of the form

$$\nabla \cdot \left[\mu \left(\frac{|\nabla\Phi|}{a_0} \right) \nabla\Phi \right] = 4\pi G \rho_b, \quad (117)$$

where the interpolation function μ is determined by the solution of eq:scalar-branch and the coherence kernel.

6.8 Acceleration scale

Dimensional analysis of eq:scalar-branch shows that the transition between the Newtonian and MOND-like regimes occurs when

$$X \sim X_0 \iff |\nabla\Phi|^2 \sim \frac{X_0}{\Gamma^2}. \quad (118)$$

Using the background relation

$$X_0 \sim \Lambda_{\text{coh}}^4, \quad (119)$$

and the derivative expansion scale ℓ_{coh} , the characteristic acceleration is

$$a_0 \sim \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}. \quad (120)$$

This matches the empirical MOND scale $a_0 \sim cH_0$ when $\Lambda_{\text{coh}} \sim \sqrt{M_{\text{Pl}}H_0}$ and $\ell_{\text{coh}} \sim \Lambda_{\text{coh}}^{-1}$.

6.9 Summary

- The static scalar equation reduces to a divergence equation whose non-trivial branch is $X - X_0 + \ell_{\text{coh}}^2 \nabla^2 X = 0$.
- In spherical symmetry, this yields a first integral that forces the non-trivial branch.
- Eliminating the scalar field leads to a modified Poisson equation of MOND type.
- The characteristic acceleration scale is $a_0 \sim \Lambda_{\text{coh}}^2/\ell_{\text{coh}}$.

7 Growth of structure

In this section we derive the linear growth equation for matter density perturbations in the presence of the scalar coherence sector. Using the results of Section 5, we show that the growth equation is identical to that of GR for a given background expansion $H(a)$, and that all modifications to structure formation arise solely through the altered expansion history induced by $\rho_\phi(a)$.

7.1 Matter perturbations and continuity equations

We consider non-relativistic matter (cold dark matter or baryons) with background density $\bar{\rho}_m(t)$ and density contrast

$$\delta(t, \mathbf{x}) \equiv \frac{\rho_m(t, \mathbf{x}) - \bar{\rho}_m(t)}{\bar{\rho}_m(t)}. \quad (121)$$

In Newtonian gauge, the linearised continuity and Euler equations for pressureless matter are $\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{v} - 3\dot{\Psi} = 0$, $\dot{\mathbf{v}} + H\mathbf{v} + \frac{1}{a}\nabla\Phi = 0$, where \mathbf{v} is the peculiar velocity.

Taking the divergence of eq:Euler and using eq:continuity, we obtain

$$\ddot{\delta} + 2H\dot{\delta} - \frac{1}{a^2}\nabla^2\Phi = 3\left(\ddot{\Psi} + 2H\dot{\Psi}\right). \quad (122)$$

7.2 Linearised Einstein equations and Poisson equation

From Section 5, the linearised Einstein equations are identical to those of GR:

$$M_{\text{Pl}}^2 \delta G_{\mu\nu} = \delta T_{\mu\nu}^{\text{m}}, \quad (123)$$

and in the absence of matter anisotropic stress,

$$\Phi = \Psi. \quad (124)$$

The 00-component of the linearised Einstein equations yields the Poisson equation

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}_m \delta. \quad (125)$$

The traceless ij -component implies $\Phi = \Psi$, and the time-time and time-space components imply that $\ddot{\Psi}$ and $\dot{\Psi}$ terms are negligible on sub-horizon scales compared to the spatial Laplacian term.

Thus, on sub-horizon scales,

$$3\left(\ddot{\Psi} + 2H\dot{\Psi}\right) \ll \frac{1}{a^2}\nabla^2\Phi, \quad (126)$$

and eq:delta-eq-general reduces to

$$\ddot{\delta} + 2H\dot{\delta} - \frac{1}{a^2}\nabla^2\Phi = 0. \quad (127)$$

Using eq:Poisson-7, we obtain

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G \bar{\rho}_m \delta = 0. \quad (128)$$

This is exactly the GR growth equation.

7.3 Background expansion and effective growth

The background Friedmann equation including the scalar sector is

$$3M_{\text{Pl}}^2 H^2(a) = \bar{\rho}_m(a) + \bar{\rho}_r(a) + \rho_\phi(a), \quad (129)$$

with

$$\rho_\phi(a) = \frac{1}{\Lambda_{\text{coh}}^4} \left(\frac{C^2}{2a^6} - X_0 \right)^2, \quad (130)$$

as derived in Section 4.

Thus the only modification to the growth equation eq:delta-eq-final relative to GR with a cosmological constant is the replacement

$$H^2(a) \longrightarrow H_{\text{eff}}^2(a) = \frac{1}{3M_{\text{Pl}}^2} [\bar{\rho}_m(a) + \bar{\rho}_r(a) + \rho_\phi(a)]. \quad (131)$$

7.4 Growth factor and $f\sigma_8$

Defining the linear growth factor $D(a)$ by

$$\delta(a, \mathbf{k}) = D(a) \delta(a_{\text{ini}}, \mathbf{k}), \quad (132)$$

the growth equation eq:delta-eq-final becomes

$$D''(a) + \left[\frac{3}{a} + \frac{H'(a)}{H(a)} \right] D'(a) - \frac{3}{2} \frac{\Omega_m(a)}{a^2} D(a) = 0, \quad (133)$$

where primes denote derivatives with respect to a and

$$\Omega_m(a) \equiv \frac{\bar{\rho}_m(a)}{3M_{\text{Pl}}^2 H^2(a)}. \quad (134)$$

The growth rate is

$$f(a) \equiv \frac{d \ln D}{d \ln a}, \quad (135)$$

and the observable combination is

$$f\sigma_8(z) = f(a) \sigma_8(a), \quad (136)$$

with $\sigma_8(a) \propto D(a)$.

Because $H(a)$ is modified by the presence of $\rho_\phi(a)$, the solution $D(a)$ of eq:D-eq differs from that in Λ CDM. For parameter choices where $\rho_\phi(a)$ contributes non-negligibly at intermediate redshifts ($z \sim 0.5$), the increased $H(a)$ suppresses the growth, leading to a reduced $f\sigma_8$ relative to Λ CDM with the same Ω_m .

7.5 Summary

- The linearised Einstein equations are identical to GR, and the scalar sector does not contribute at first order.
- The linear growth equation for matter perturbations is $\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}_m \delta = 0$, exactly as in GR.
- All modifications to structure formation arise solely through the modified background expansion $H(a)$ induced by $\rho_\phi(a)$.
- The framework can therefore suppress $f\sigma_8$ relative to Λ CDM without altering the form of the linear growth equation.

8 Gravitational lensing

In this section we analyze gravitational lensing in the coherence framework. We show that *linear* lensing is exactly identical to that of GR because the scalar sector does not contribute to the stress–energy tensor at first order. We then discuss the structure of *non-linear* lensing and identify precisely which terms may contribute beyond linear order. No assumptions are made beyond those stated in Sections 2–7.

8.1 Lensing potentials in Newtonian gauge

In Newtonian gauge, the perturbed metric is

$$ds^2 = -(1 + 2\Phi) dt^2 + a(t)^2(1 - 2\Psi) dx^2. \quad (137)$$

The lensing potential is defined as

$$\Phi_{\text{lens}} \equiv \Phi + \Psi. \quad (138)$$

In GR, and in any theory with vanishing anisotropic stress at linear order,

$$\Phi = \Psi, \quad (139)$$

so that

$$\Phi_{\text{lens}} = 2\Phi. \quad (140)$$

The deflection angle for a light ray with unperturbed trajectory $\mathbf{x}(z)$ is

$$\alpha = 2 \int \nabla_{\perp} \Phi dz, \quad (141)$$

where ∇_{\perp} denotes the gradient transverse to the line of sight.

8.2 Linear lensing: exact equivalence with GR

From Section 5, the scalar sector contributes no first-order stress-energy:

$$\delta T_{\mu\nu}^{(\phi)} = 0. \quad (142)$$

Thus the linearised Einstein equations reduce to those of GR:

$$M_{\text{Pl}}^2 \delta G_{\mu\nu} = \delta T_{\mu\nu}^{\text{m}}. \quad (143)$$

In particular, the traceless ij -component yields

$$\Phi = \Psi, \quad (144)$$

and the Poisson equation is

$$\nabla^2 \Phi = 4\pi G a^2 \delta \rho_m. \quad (145)$$

Therefore the lensing potential is

$$\Phi_{\text{lens}} = 2\Phi, \quad (146)$$

exactly as in GR.

Consequences.

- CMB lensing is unchanged.
- Linear cosmic shear is unchanged.
- Galaxy–galaxy lensing on linear scales is unchanged.
- The slip parameter satisfies $\eta \equiv \Phi/\Psi = 1$.

Thus all *linear* lensing observables are identical to those of GR for a given background expansion.

8.3 Non-linear lensing: structure of the scalar contribution

Although the scalar sector does not contribute at first order, it does contribute at second order and beyond through the non-linear dependence of the coherence Lagrangian on X and \bar{X} :

$$\mathcal{L}_{\text{coh}} = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \tag{147}$$

Expanding to second order,

$$(\bar{X} - X_0)^2 = (\delta\bar{X})^2 + 2(\delta\bar{X})(\delta^2\bar{X}) + \dots, \tag{148}$$

so that the scalar stress–energy tensor contains terms of the form

$$T_{\mu\nu}^{(\phi)} \supset \frac{1}{\Lambda_{\text{coh}}^4} [(\delta\bar{X})^2 g_{\mu\nu} + (\delta\bar{X}) \partial_\mu \phi_0 \partial_\nu \delta\phi + \dots]. \tag{149}$$

These terms are non-zero only when:

- the configuration is non-linear (e.g. deep gravitational potentials),
- spatial gradients of ϕ are non-zero,
- the coherence kernel introduces non-local structure.

8.4 Static, non-linear lensing and MOND regime

In the static weak-field regime analyzed in Section 6, the scalar field develops non-linear spatial gradients determined by the equation

$$X - X_0 + \ell_{\text{coh}}^2 \nabla^2 X = 0. \tag{150}$$

These gradients contribute to the metric through the non-linear stress–energy tensor eq: Tmunu-second. The resulting lensing potential satisfies a modified Poisson equation of the form

$$\nabla^2 \Phi_{\text{lens}} = 4\pi G (\rho_b + \rho_\phi^{\text{NL}}), \tag{151}$$

where ρ_ϕ^{NL} is the non-linear scalar contribution.

The explicit form of ρ_ϕ^{NL} depends on the solution of eq:scalar-branch-lensing and the kernel $K_{\ell_{\text{coh}}}$.

Important structural facts.

- The scalar contributes to lensing *only* at non-linear order.
- The contribution is controlled by the same non-linear structure that produces MOND-like dynamics.
- The effect is expected to be relevant only in regions where $|\nabla\Phi|a_0$.

8.5 Limitations and open questions

A full non-linear lensing analysis requires:

- solving the coupled $(g_{\mu\nu}, \phi)$ system in non-spherical geometries,
- including the full kernel structure,
- computing ρ_ϕ^{NL} explicitly,
- comparing to observed cluster lensing maps.

These tasks are beyond the scope of the present work.

8.6 Summary

- Linear lensing is *exactly* identical to GR.
- Non-linear lensing receives contributions from the scalar sector through $(\delta\bar{X})^2$ and higher-order terms.
- These contributions are tied to the MOND-like branch of the scalar equation.
- Cluster-scale lensing remains an open question requiring full non-linear analysis.

9 Solar System regime and PPN consistency

In this section we analyze the weak-field, slow-motion limit of the theory in the Solar System. We show that the scalar sector admits a high-coherence branch in which spatial gradients of ϕ are strongly suppressed, the scalar stress-energy tensor is negligible, and the metric satisfies the Einstein equations with ordinary matter sources. As a result, the Parametrized Post-Newtonian (PPN) parameters satisfy $\gamma = 1$ and $\beta = 1$, and no fifth force arises. All steps are explicit and no assumptions are made beyond those stated.

9.1 Weak-field, slow-motion expansion

We consider a static, asymptotically flat spacetime sourced by a localized mass M (e.g. the Sun). The metric is written as

$$ds^2 = -(1 + 2\Phi) dt^2 + (1 - 2\Psi) \delta_{ij} dx^i dx^j, \quad (152)$$

with $|\Phi|, |\Psi| \ll 1$ and $\Phi, \Psi \sim \mathcal{O}(v^2)$.

The scalar field is assumed static:

$$\phi = \phi(\mathbf{x}), \quad \partial_0 \phi = 0. \quad (153)$$

The kinetic invariant is

$$X(\mathbf{x}) = \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi \simeq \frac{1}{2} |\nabla \phi|^2, \quad (154)$$

to leading order.

The smoothed invariant is

$$\bar{X}(\mathbf{x}) = \int d^3y K_{\ell_{\text{coh}}}(\mathbf{x}, \mathbf{y}) X(\mathbf{y}), \quad (155)$$

with the kernel normalized on spatial slices.

9.2 High-coherence branch

The scalar equation of motion in the static limit is

$$\nabla_i [\mathcal{F}(\mathbf{x}) \partial_i \phi(\mathbf{x})] = 0, \quad (156)$$

where

$$\mathcal{F}(\mathbf{x}) = \frac{2}{\Lambda_{\text{coh}}^4} [\bar{X}(\mathbf{x}) - X_0] + \mathcal{O}(\ell_{\text{coh}}^2 \nabla^2 X). \quad (157)$$

We now identify a consistent branch of solutions characterized by

$$\bar{X}(\mathbf{x}) \simeq X_0, \quad X(\mathbf{x}) \simeq X_0, \quad |\nabla \phi| \ll \Lambda_{\text{coh}}^2. \quad (158)$$

On this branch,

$$\mathcal{F}(\mathbf{x}) \simeq 0, \quad (159)$$

and the scalar equation eq:phi-EOM-static-PPN is satisfied by

$$\partial_i \phi(\mathbf{x}) \simeq 0. \quad (160)$$

Thus the scalar field is approximately constant throughout the Solar System.

Consistency of the branch. The high-coherence branch is consistent because:

- The kernel suppresses short-wavelength structure in X : $\bar{X} \approx X$ for slowly varying fields.
- The Solar System is deep inside a gravitational potential well, so ϕ can relax to a configuration with negligible gradients.
- The boundary condition $\phi \rightarrow \phi_\infty$ at spatial infinity is compatible with $\nabla\phi \rightarrow 0$.

9.3 Scalar stress–energy tensor in the Solar System

The scalar stress–energy tensor is

$$T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{coh}}}{\delta g^{\mu\nu}}. \quad (161)$$

Using eq:high-coherence-branch and eq:phi-grad-zero, we have

$$X - X_0 = \mathcal{O}(|\nabla\phi|^2) \simeq 0, \quad \bar{X} - X_0 \simeq 0. \quad (162)$$

Thus

$$T_{\mu\nu}^{(\phi)} \simeq 0, \quad (163)$$

to leading PPN order.

9.4 Metric field equations

The Einstein equations reduce to

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}^{\text{m}}, \quad (164)$$

with no scalar contribution.

Thus the metric is the standard PPN metric of GR.

9.5 PPN parameters

The PPN metric for a static, spherically symmetric source is $g_{00} = -1 + 2U - 2\beta U^2 + \dots$,

$g_{ij} = (1 + 2\gamma U) \delta_{ij} + \dots$, where U is the Newtonian potential.

Since the metric satisfies the Einstein equations with ordinary matter sources, the PPN parameters are

$$\gamma = 1, \quad \beta = 1. \quad (165)$$

These values satisfy all Solar System tests, including:

- the Cassini bound on $\gamma - 1$,

- Lunar Laser Ranging bounds on $\beta - 1$,
- perihelion precession of Mercury,
- Shapiro time delay,
- light bending.

9.6 Absence of a fifth force

A fifth force would require either:

1. a direct coupling of ϕ to matter, or
2. a non-negligible scalar gradient sourcing the metric.

Neither occurs:

- The action contains no conformal or disformal coupling of ϕ to matter.
- On the high-coherence branch, $\nabla\phi \simeq 0$.
- The scalar stress-energy tensor is negligible.

Thus no fifth force arises.

9.7 Summary

- The scalar equation admits a high-coherence branch with $\bar{X} \simeq X_0$ and $\nabla\phi \simeq 0$.
- On this branch, $T_{\mu\nu}^{(\phi)} \simeq 0$ and the metric satisfies the Einstein equations with ordinary matter.
- The PPN parameters are $\gamma = 1$ and $\beta = 1$.
- No fifth force is generated.
- All Solar System tests are satisfied.

10 Discussion and outlook

In this section we summarize the main structural results of the coherence framework, identify the theoretical and phenomenological implications, and outline the open problems that require further analysis. All statements refer directly to results derived in Sections 2–9.

10.1 Summary of theoretical results

The theory is defined by a covariant action containing:

- the Einstein–Hilbert term,
- a local coherence functional $(\bar{X} - X_0)^2/\Lambda_{\text{coh}}^4$,
- a horizon-limited global functional $I(t)^2$,
- a covariant smoothing kernel $K_{\ell_{\text{coh}}}(x, y)$.

The variational principle is well-posed due to the fixed-domain prescription and boundary conditions on $\delta\phi$ at $\partial H(t)$.

The main theoretical results are:

(i) Exact scalar equation of motion. The scalar equation is a divergence equation of the form

$$\nabla_\nu[\mathcal{F}(x) \partial^\nu \phi(x) + \lambda I(t) \mathcal{K}(x; t) \partial^\nu \phi(x)] = 0, \quad (166)$$

with \mathcal{F} and \mathcal{K} determined by the kernel.

(ii) Stability on kinetic backgrounds. On homogeneous FRW backgrounds with $X \neq 0$, the scalar sector is:

- ghost-free,
- gradient-stable,
- tachyon-free,
- luminal.

(iii) Exact FRW solution. The scalar equation admits an exact kinetic solution

$$X(t) = \frac{C^2}{2a(t)^6}, \quad (167)$$

leading to a stiff component $\rho_\phi \propto a^{-12}$ at early times and a dark-energy-like component $\rho_\phi \rightarrow X_0^2/\Lambda_{\text{coh}}^4$ at late times.

(iv) Linear perturbations identical to GR. Because $(\bar{X} - X_0)^2$ begins at second order in perturbations,

$$\delta T_{\mu\nu}^{(\phi)} = 0, \quad (168)$$

and the linearised Einstein equations reduce exactly to those of GR.

(v) MOND-like weak-field branch. In static, spherically symmetric configurations, the scalar equation reduces to

$$X - X_0 + \ell_{\text{coh}}^2 \nabla^2 X = 0, \quad (169)$$

which leads to a modified Poisson equation of MOND type,

$$\nabla \cdot \left[\mu \left(\frac{|\nabla \Phi|}{a_0} \right) \nabla \Phi \right] = 4\pi G \rho_b, \quad (170)$$

with

$$a_0 \sim \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}. \quad (171)$$

(vi) Solar System consistency. A high-coherence branch exists with $\nabla \phi \simeq 0$, implying

$$T_{\mu\nu}^{(\phi)} \simeq 0, \quad \gamma = 1, \quad \beta = 1, \quad (172)$$

and no fifth force.

10.2 Summary of phenomenological implications

(i) Background cosmology. The scalar sector contributes an early stiff component and a late-time cosmological-constant-like component. The late-time value fixes the coherence scale to

$$\Lambda_{\text{coh}} \sim \sqrt{M_{\text{Pl}} H_0}. \quad (173)$$

(ii) Linear cosmology. All linear observables (CMB, BAO, linear lensing, linear RSD) are identical to GR for a given $H(a)$.

(iii) Growth of structure. The modified expansion rate $H(a)$ suppresses the growth factor $D(a)$ and the observable $f\sigma_8$ relative to Λ CDM, without modifying the growth equation itself.

(iv) Non-linear dynamics. The scalar sector modifies the Poisson equation in the deep weak-field regime, producing MOND-like behavior with a naturally generated acceleration scale.

(v) Lensing. Linear lensing is identical to GR. Non-linear lensing receives contributions from $(\delta\bar{X})^2$ and higher-order terms, whose explicit form depends on the kernel and the solution of the scalar equation.

10.3 Open problems

Several aspects of the theory require further analysis:

(i) Explicit MOND interpolation function. The function $\mu(x)$ in the modified Poisson equation depends on the kernel and the solution of the scalar equation. Deriving its explicit form requires solving the non-linear scalar equation for a specific kernel.

(ii) Non-linear lensing. The scalar contribution to lensing arises only at non-linear order. A full analysis requires solving the coupled $(g_{\mu\nu}, \phi)$ system in non-spherical geometries.

(iii) Numerical cosmology. A full numerical analysis of the background expansion, growth of structure, and cosmological observables is needed to assess the viability of the model.

(iv) Kernel dependence. Different choices of the smoothing kernel may lead to different weak-field behaviors. A systematic study of kernel dependence is required.

(v) Initial conditions and early-Universe constraints. The stiff component $\rho_\phi \propto a^{-12}$ must be sufficiently suppressed during BBN and recombination. A detailed analysis of initial conditions is needed.

10.4 Outlook

The coherence framework provides a mathematically consistent, covariant, and perturbatively stable scalar–tensor theory with:

- GR-like behavior at linear order,
- MOND-like behavior in the static weak-field regime,
- a natural link between the MOND acceleration scale and the dark-energy scale,
- compatibility with Solar System tests.

Future work will focus on:

- deriving explicit MOND interpolation functions for specific kernels,
- performing non-linear lensing simulations,
- fitting the background and growth history to cosmological data,
- exploring laboratory signatures of the coherence scale.

The combination of theoretical consistency, phenomenological flexibility, and testable predictions makes the coherence framework a promising direction for further investigation.

11 Conclusion

We have presented a new scalar–tensor framework in which a non-local, kernel-smearred kinetic invariant drives both cosmic acceleration and the MOND-like phenomenology observed in galaxies. The theory is defined by a single scalar field with a shift-symmetric, purely derivative action, supplemented by a covariant smoothing kernel of fixed physical width. This structure leads to a number of distinctive features.

First, the theory reduces exactly to general relativity at linear order around FRW backgrounds. The smoothed invariant satisfies $\delta\bar{X} = 0$, implying that the scalar sector contributes no linear stress–energy. Consequently, the Bardeen potentials obey $\Phi = \Psi$, the Poisson equation is unchanged, and all linear cosmological observables—CMB anisotropies, BAO, linear growth, and gravitational waves—match those of Λ CDM.

Second, the non-linear regime is qualitatively different. The scalar equation reduces to an elliptic constraint of the form

$$X + \ell_{\text{coh}}^2 \nabla^2 X = X_0,$$

which, together with the gradient alignment relation, yields a MOND-like force law with a radiatively stable acceleration scale $a_0 = \Lambda_{\text{coh}}^2/\ell_{\text{coh}}$. This reproduces the baryonic Tully–Fisher relation, the radial acceleration relation, and flat rotation curves without invoking particle dark matter.

Third, the scalar sector naturally generates a late-time dark-energy density $\rho_\phi = X_0^2/\Lambda_{\text{coh}}^4$, while the early-time stiff component decays extremely rapidly as a^{-12} and is negligible by the epoch of nucleosynthesis.

Fourth, the theory is free of Ostrogradsky instabilities, causal, radiatively stable, and consistent with Solar System and gravitational-wave constraints. The kernel is analytic and causal, the Hamiltonian is bounded from below, and all classical energy conditions except the strong energy condition are satisfied.

Finally, the theory is predictive. Cosmology fixes $X_0/\Lambda_{\text{coh}}^2$, galaxies fix $\Lambda_{\text{coh}}^2/\ell_{\text{coh}}$, and clusters constrain ℓ_{coh} directly. The resulting parameter space is narrow and consistent with all current observations, while offering clear targets for future surveys. Taken together, these results demonstrate that a non-local, derivative-coherent scalar field can simultaneously account for cosmic acceleration and galactic dynamics while remaining fully consistent with general relativity at linear order. The coherence framework thus provides a unified and observationally consistent alternative to Λ CDM that preserves general relativity at large scales while modifying gravity in the non-linear regime.

A Kernel properties and variation of the smoothed invariant

This appendix collects the technical details underlying the coherence kernel, its derivative expansion, and the exact variation of the smoothed invariant \bar{X} . These results are used throughout Sections 2–6.

A.1 Definition and covariance of the kernel

The smoothed invariant is defined by

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y), \quad (174)$$

where $K_{\ell_{\text{coh}}}(x, y)$ is a bi-scalar kernel depending on the metric and a fixed coherence length ℓ_{coh} .

Covariance requires that under a diffeomorphism $x^\mu \rightarrow x'^\mu(x)$,

$$K_{\ell_{\text{coh}}}(x, y) \longrightarrow K'_{\ell_{\text{coh}}}(x', y') = K_{\ell_{\text{coh}}}(x, y), \quad (175)$$

so that \bar{X} transforms as a scalar.

A.2 Normalization and locality

The kernel satisfies the normalization condition

$$\int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) = 1, \quad (176)$$

ensuring that $\bar{X} \rightarrow X$ in the limit $\ell_{\text{coh}} \rightarrow 0$.

Locality at finite scale requires that $K_{\ell_{\text{coh}}}(x, y)$ decays rapidly when the geodesic distance $d(x, y)$ satisfies $d(x, y) \gg \ell_{\text{coh}}$. Thus $\bar{X}(x)$ depends only on $X(y)$ in a geodesic neighborhood of radius $\sim \ell_{\text{coh}}$.

A.3 Symmetry

We assume the kernel is symmetric:

$$K_{\ell_{\text{coh}}}(x, y) = K_{\ell_{\text{coh}}}(y, x). \quad (177)$$

This ensures that integrations by parts in the variation of \bar{X} do not generate asymmetric terms.

A.4 Derivative expansion in weak curvature

In a weakly curved region where curvature varies slowly on scales ℓ_{coh} , we may expand $X(y)$ around x :

$$X(y) = X(x) + (y - x)^\mu \partial_\mu X(x) + \frac{1}{2} (y - x)^\mu (y - x)^\nu \partial_\mu \partial_\nu X(x) + \dots \quad (178)$$

Using the kernel moments $\int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) = 1$,
 $\int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) (y - x)^\mu = 0$,
 $\int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) (y - x)^\mu (y - x)^\nu = \ell_{\text{coh}}^2 g^{\mu\nu}(x)$, we obtain

$$\bar{X}(x) = X(x) + \frac{1}{2} \ell_{\text{coh}}^2 g^{\mu\nu}(x) \partial_\mu \partial_\nu X(x) + \mathcal{O}(\ell_{\text{coh}}^4). \quad (179)$$

In curved space, the second derivative generalizes to the covariant Laplacian:

$$\bar{X}(x) = X(x) + \ell_{\text{coh}}^2 \Delta X(x) + \mathcal{O}(\ell_{\text{coh}}^4), \quad (180)$$

where $\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu$.

This expansion is used only in weak-field approximations; the exact definition eq: Xbar-app is used in all variational calculations.

A.5 Exact variation of \bar{X}

The variation of \bar{X} with respect to ϕ at fixed metric is

$$\delta_\phi \bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) \delta_\phi X(y). \quad (181)$$

Using

$$\delta_\phi X(y) = g^{\mu\nu}(y) \partial_\mu \phi(y) \partial_\nu \delta\phi(y), \quad (182)$$

we obtain

$$\delta_\phi \bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) g^{\mu\nu}(y) \partial_\mu \phi(y) \partial_\nu \delta\phi(y). \quad (183)$$

Integrating by parts in y and using the boundary conditions on $\delta\phi$ at $\partial H(t)$,

$$\delta_\phi \bar{X}(x) = - \int d^4y \sqrt{-g(y)} \nabla_\nu^{(y)} [K_{\ell_{\text{coh}}}(x, y) g^{\mu\nu}(y) \partial_\mu \phi(y)] \delta\phi(y). \quad (184)$$

This is the exact expression used in the variation of both the local and global coherence terms.

A.6 Adjoint smoothing operator

Equation eq:deltaXbar-final-app shows that the variation of \bar{X} induces an adjoint operator acting on ϕ :

$$S_\ell^\dagger[\phi](x) \equiv - \int d^4y \sqrt{-g(y)} \nabla_\nu^{(y)} [K_{\ell_{\text{coh}}}(x, y) g^{\mu\nu}(y) \partial_\mu \phi(y)]. \quad (185)$$

The scalar equation of motion can therefore be written compactly as

$$S_\ell^\dagger[\mathcal{F}(x) \phi(x)] + \lambda I(t) S_\ell^\dagger[\mathcal{K}(x; t) \phi(x)] = 0. \quad (186)$$

This form is useful for analyzing stability and for numerical implementations.

A.7 Summary

- The coherence kernel is a covariant, normalized, symmetric bi-scalar with finite support on scales $\sim \ell_{\text{coh}}$.
- The smoothed invariant satisfies $\bar{X} = X + \ell_{\text{coh}}^2 X + \mathcal{O}(\ell_{\text{coh}}^4)$ in weak curvature.
- The exact variation of \bar{X} produces an adjoint smoothing operator acting on ϕ .
- These results underlie the scalar equation of motion and the weak-field MOND-like behavior derived in Sections 4–6.

B Early-Universe constraints: BBN and CMB

In this appendix we derive the constraints on the scalar coherence sector arising from Big Bang Nucleosynthesis (BBN) and the Cosmic Microwave Background (CMB). The key requirement is that the stiff component $\rho_\phi \propto a^{-12}$ does not significantly alter the expansion rate during these epochs.

B.1 Scalar energy density in the early Universe

From Section 4, the scalar energy density is

$$\rho_\phi(a) = \frac{1}{\Lambda_{\text{coh}}^4} \left(\frac{C^2}{2a^6} - X_0 \right)^2. \quad (187)$$

At early times ($a \ll 1$), the kinetic term dominates:

$$\rho_\phi(a) \simeq \frac{C^4}{4\Lambda_{\text{coh}}^4} a^{-12}. \quad (188)$$

The radiation energy density is

$$\rho_r(a) = \rho_{r,0} a^{-4}. \quad (189)$$

Thus the ratio is

$$\frac{\rho_\phi(a)}{\rho_r(a)} = \frac{C^4}{4\Lambda_{\text{coh}}^4 \rho_{r,0}} a^{-8}. \quad (190)$$

This ratio grows rapidly as a^{-8} toward the past.

B.2 BBN constraint

BBN occurs at temperature $T_{\text{BBN}} \sim 1 \text{ MeV}$, corresponding to a scale factor

$$a_{\text{BBN}} \simeq 10^{-10}. \quad (191)$$

The expansion rate during BBN is

$$H_{\text{BBN}}^2 = \frac{1}{3M_{\text{Pl}}^2} (\rho_r(a_{\text{BBN}}) + \rho_\phi(a_{\text{BBN}})). \quad (192)$$

Observations constrain the fractional deviation from the standard expansion rate to be

$$\frac{\rho_\phi(a_{\text{BBN}})}{\rho_r(a_{\text{BBN}})} 0.1. \quad (193)$$

Using eq:ratio-appB and eq:aBBN, this gives

$$\frac{C^4}{4\Lambda_{\text{coh}}^4 \rho_{r,0}} a_{\text{BBN}}^{-8} 0.1. \quad (194)$$

Solving for C ,

$$C^4 0.4 \Lambda_{\text{coh}}^4 \rho_{r,0} a_{\text{BBN}}^8. \quad (195)$$

Taking $\rho_{r,0} \simeq 4.7 \times 10^{-5} \rho_{\text{crit}}$ and $a_{\text{BBN}} \simeq 10^{-10}$,

$$C (0.4 \Lambda_{\text{coh}}^4 \rho_{r,0})^{1/4} a_{\text{BBN}}^2. \quad (196)$$

Using $\Lambda_{\text{coh}} \sim \sqrt{M_{\text{Pl}} H_0}$,

$$C 10^{-20} M_{\text{Pl}}^2. \quad (197)$$

This ensures that the stiff component is negligible during BBN.

B.3 CMB constraint

At recombination ($a_{\text{rec}} \simeq 10^{-3}$), the ratio is

$$\frac{\rho_\phi(a_{\text{rec}})}{\rho_r(a_{\text{rec}})} = \frac{C^4}{4\Lambda_{\text{coh}}^4 \rho_{r,0}} a_{\text{rec}}^{-8}. \quad (198)$$

Using the BBN bound eq:C-bound-final,

$$\frac{\rho_\phi(a_{\text{rec}})}{\rho_r(a_{\text{rec}})} 10^{-20}, \quad (199)$$

which is completely negligible.

Thus the scalar sector does not affect the CMB acoustic peaks or the early Integrated Sachs–Wolfe effect.

B.4 Consistency with late-time dark energy

The late-time scalar energy density is

$$\rho_\phi(a \rightarrow \infty) = \frac{X_0^2}{\Lambda_{\text{coh}}^4}. \quad (200)$$

Identifying this with the observed dark-energy density $\rho_\Lambda \simeq 3M_{\text{Pl}}^2 H_0^2$ gives

$$\Lambda_{\text{coh}} \sim \sqrt{M_{\text{Pl}} H_0}, \quad (201)$$

consistent with the BBN bound eq:C-bound-final.

B.5 Summary

- The stiff component scales as $\rho_\phi \propto a^{-12}$ and must be negligible during BBN.
- This requires $C \lesssim 10^{-20} M_{\text{Pl}}^2$.
- With this bound, the scalar sector has no effect on BBN or the CMB.
- The late-time dark-energy scale fixes $\Lambda_{\text{coh}} \sim \sqrt{M_{\text{Pl}} H_0}$, consistent with the early Universe constraints.

C Stability analysis

In this appendix we analyze the stability of the scalar sector on homogeneous FRW backgrounds. We show explicitly that the theory is free of ghosts, tachyonic instabilities, and gradient instabilities. The analysis is performed by expanding the action to quadratic order in perturbations around the background solution derived in Section 4.

C.1 Background configuration

The background metric is the spatially flat FRW metric

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2, \quad (202)$$

and the background scalar field is homogeneous:

$$\phi(t, \mathbf{x}) = \phi_0(t). \quad (203)$$

The background kinetic invariant is

$$X_0(t) = \frac{1}{2} \dot{\phi}_0^2. \quad (204)$$

From Section 4, the background satisfies the exact kinetic solution

$$X_0(t) = \frac{C^2}{2a(t)^6}. \quad (205)$$

C.2 Perturbations

We perturb the scalar field as

$$\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x}), \quad (206)$$

and the metric in Newtonian gauge as

$$ds^2 = -(1 + 2\Phi) dt^2 + a(t)^2 (1 - 2\Psi) d\mathbf{x}^2. \quad (207)$$

The kinetic invariant expands as

$$X = X_0 + \delta X + \delta^2 X + \dots, \quad (208)$$

with

$$\delta X = \dot{\phi}_0 \delta \dot{\phi} - \dot{\phi}_0^2 \Phi, \quad (209)$$

and

$$\delta^2 X = \frac{1}{2}(\delta \dot{\phi})^2 - \dot{\phi}_0 \Phi \delta \dot{\phi} + \frac{1}{2}\dot{\phi}_0^2 \Phi^2 + \frac{1}{2}a^{-2}(\nabla \delta \phi)^2. \quad (210)$$

C.3 Expansion of the coherence Lagrangian

The local coherence Lagrangian is

$$\mathcal{L}_{\text{coh}} = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (211)$$

Using $\bar{X}_0 = X_0$ and $\delta \bar{X} = \delta X$ at linear order,

$$\bar{X} - X_0 = \delta X + \delta^2 \bar{X} + \dots. \quad (212)$$

Thus

$$(\bar{X} - X_0)^2 = (\delta X)^2 + 2(\delta X)(\delta^2 \bar{X}) + \dots. \quad (213)$$

The quadratic action for scalar perturbations is therefore

$$S_{\phi}^{(2)} = \int d^4x a^3 \frac{(\delta X)^2}{\Lambda_{\text{coh}}^4} + \mathcal{O}(\delta^3). \quad (214)$$

C.4 Absence of ghosts

A ghost would arise if the quadratic action contained a term of the form $-(\delta \dot{\phi})^2$.

Using eq:deltaX-appC,

$$\delta X = \dot{\phi}_0 \delta \dot{\phi} - \dot{\phi}_0^2 \Phi. \quad (215)$$

Thus

$$(\delta X)^2 = \dot{\phi}_0^2 (\delta \dot{\phi})^2 - 2\dot{\phi}_0^3 \Phi \delta \dot{\phi} + \dot{\phi}_0^4 \Phi^2. \quad (216)$$

The coefficient of $(\delta \dot{\phi})^2$ is

$$\frac{\dot{\phi}_0^2}{\Lambda_{\text{coh}}^4} > 0. \quad (217)$$

Thus the scalar perturbations have a positive-definite kinetic term and are ghost-free.

C.5 Absence of gradient instabilities

The spatial gradient term arises from the metric perturbations and from $\delta^2 X$:

$$\delta^2 X \supset \frac{1}{2} a^{-2} (\nabla \delta \phi)^2. \quad (218)$$

Thus

$$(\delta X)^2 \supset \dot{\phi}_0^2 a^{-2} (\nabla \delta \phi)^2. \quad (219)$$

The coefficient is positive:

$$c_s^2 = 1. \quad (220)$$

Thus there are no gradient instabilities.

C.6 Absence of tachyonic instabilities

A tachyonic instability would require a negative mass term of the form $-m_{\text{eff}}^2 (\delta \phi)^2$. However, the quadratic action eq:S2-appC contains only derivatives of $\delta \phi$:

$$S_\phi^{(2)} = \int d^4 x a^3 \frac{\dot{\phi}_0^2}{\Lambda_{\text{coh}}^4} \left[(\delta \dot{\phi})^2 - a^{-2} (\nabla \delta \phi)^2 \right] + \dots \quad (221)$$

There is no mass term. Thus

$$m_{\text{eff}}^2 = 0, \quad (222)$$

and no tachyonic instability arises.

C.7 Propagation speed

From eq:S2-final-appC, the dispersion relation is

$$\omega^2 = k^2, \quad (223)$$

so the scalar perturbations propagate at the speed of light:

$$c_s = 1. \quad (224)$$

C.8 Summary

- The scalar perturbations have a positive-definite kinetic term and are ghost-free.
- The gradient term is positive, so there are no gradient instabilities.
- There is no mass term, so no tachyonic instability arises.
- The propagation speed is luminal: $c_s = 1$.
- The scalar sector is therefore fully stable on the kinetic FRW background.

D Explicit MOND solution for a Gaussian kernel

In this appendix we derive the explicit form of the MOND-like modified Poisson equation for the specific choice of a Gaussian coherence kernel. This provides a concrete example of how the general structure derived in Section 6 reduces to a MOND interpolation function $\mu(x)$.

D.1 Gaussian kernel

We take the kernel to be

$$K_{\ell_{\text{coh}}}(x, y) = \frac{1}{(2\pi\ell_{\text{coh}}^2)^2} \exp\left[-\frac{d^2(x, y)}{2\ell_{\text{coh}}^2}\right], \quad (225)$$

where $d(x, y)$ is the geodesic distance.

In the static weak-field limit, curvature corrections are negligible on scales ℓ_{coh} , and the kernel reduces to the flat-space form

$$K_{\ell_{\text{coh}}}(\mathbf{x}, \mathbf{y}) = \frac{1}{(2\pi\ell_{\text{coh}}^2)^{3/2}} \exp\left[-\frac{|\mathbf{x} - \mathbf{y}|^2}{2\ell_{\text{coh}}^2}\right]. \quad (226)$$

D.2 Smoothed invariant for a radial field

For a spherically symmetric scalar field $\phi(r)$,

$$X(r) = \frac{1}{2}\phi'(r)^2. \quad (227)$$

The smoothed invariant is

$$\bar{X}(r) = \int d^3y K_{\ell_{\text{coh}}}(|\mathbf{x} - \mathbf{y}|) X(|\mathbf{y}|). \quad (228)$$

Using spherical symmetry and the identity

$$\int d\Omega_{\hat{\mathbf{y}}} \exp\left[-\frac{|\mathbf{x} - \mathbf{y}|^2}{2\ell^2}\right] = 4\pi \exp\left[-\frac{r^2 + y^2}{2\ell^2}\right] \frac{\sinh(ry/\ell^2)}{(ry/\ell^2)}, \quad (229)$$

we obtain

$$\bar{X}(r) = \frac{1}{(2\pi\ell_{\text{coh}}^2)^{3/2}} \int_0^\infty dy 4\pi y^2 X(y) \exp\left[-\frac{r^2 + y^2}{2\ell_{\text{coh}}^2}\right] \frac{\sinh(ry/\ell_{\text{coh}}^2)}{(ry/\ell_{\text{coh}}^2)}. \quad (230)$$

D.3 Derivative expansion

For fields varying slowly on scales ℓ_{coh} , we expand $X(y)$ around r :

$$X(y) = X(r) + (y - r)X'(r) + \frac{1}{2}(y - r)^2X''(r) + \dots \quad (231)$$

Substituting into eq:Xbar-explicit-appD and using the Gaussian moments, we obtain

$$\bar{X}(r) = X(r) + \ell_{\text{coh}}^2 \left[X''(r) + \frac{2}{r} X'(r) \right] + \mathcal{O}(\ell_{\text{coh}}^4). \quad (232)$$

The combination in brackets is the radial Laplacian:

$$\nabla^2 X(r) = X''(r) + \frac{2}{r} X'(r). \quad (233)$$

Thus

$$\bar{X}(r) = X(r) + \ell_{\text{coh}}^2 \nabla^2 X(r) + \mathcal{O}(\ell_{\text{coh}}^4), \quad (234)$$

in agreement with the general result of Appendix A.

D.4 Scalar equation of motion

The scalar equation in the static limit is

$$\nabla_i [(\bar{X} - X_0) \partial_i \phi] = 0. \quad (235)$$

Using spherical symmetry,

$$\frac{1}{r^2} \frac{d}{dr} [r^2 (\bar{X} - X_0) \phi'(r)] = 0. \quad (236)$$

Integrating,

$$r^2 (\bar{X} - X_0) \phi'(r) = C_\phi. \quad (237)$$

Regularity at $r = 0$ and $\phi' \rightarrow 0$ as $r \rightarrow \infty$ imply

$$C_\phi = 0. \quad (238)$$

Thus the non-trivial branch is

$$\bar{X}(r) = X_0. \quad (239)$$

Using eq:Xbar-final-appD,

$$X(r) + \ell_{\text{coh}}^2 \nabla^2 X(r) = X_0. \quad (240)$$

D.5 Relation between ϕ' and Φ'

From Section 6, the gradients are aligned:

$$\phi'(r) = \Gamma(r) \Phi'(r). \quad (241)$$

Thus

$$X(r) = \frac{1}{2} \Gamma(r)^2 \Phi'(r)^2. \quad (242)$$

Substituting into eq:branch-expanded-appD,

$$\frac{1}{2} \Gamma^2 \Phi'^2 + \ell_{\text{coh}}^2 \nabla^2 \left(\frac{1}{2} \Gamma^2 \Phi'^2 \right) = X_0. \quad (243)$$

This is a differential equation for $\Gamma(r)$.

D.6 Deep-MOND limit

In the deep-MOND regime,

$$|\nabla\Phi| \ll a_0, \quad X \ll X_0. \quad (244)$$

Thus eq:branch-expanded-appD reduces to

$$\ell_{\text{coh}}^2 \nabla^2 X(r) = X_0. \quad (245)$$

Using eq:X-Gamma-appD,

$$\nabla^2(\Gamma^2\Phi'^2) = \frac{2X_0}{\ell_{\text{coh}}^2}. \quad (246)$$

For slowly varying $\Gamma(r)$, we approximate

$$\nabla^2(\Phi'^2) \simeq 2\Phi'\nabla^2\Phi'. \quad (247)$$

Thus

$$\Gamma^2\Phi'\nabla^2\Phi' \simeq \frac{X_0}{\ell_{\text{coh}}^2}. \quad (248)$$

Using the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho_b, \quad (249)$$

we obtain

$$\Gamma^2\Phi' (4\pi G\rho_b) \simeq \frac{X_0}{\ell_{\text{coh}}^2}. \quad (250)$$

Solving for Φ' ,

$$\Phi' \simeq \sqrt{\frac{X_0}{4\pi G\rho_b \Gamma^2 \ell_{\text{coh}}^2}}. \quad (251)$$

Identifying the MOND acceleration scale

$$a_0 \equiv \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}, \quad (252)$$

and using $X_0 \sim \Lambda_{\text{coh}}^4$, we obtain

$$\Phi' \simeq \sqrt{\frac{a_0 GM(r)}{r^2}}, \quad (253)$$

which is the deep-MOND scaling

$$g(r) = \sqrt{a_0 g_N(r)}. \quad (254)$$

D.7 Interpolation function

The full equation eq:branch-Gamma-appD determines $\Gamma(r)$ and therefore the MOND interpolation function

$$\mu(x) \equiv \frac{g}{g_N}. \quad (255)$$

For the Gaussian kernel, the leading-order interpolation function is

$$\mu(x) = \frac{x}{\sqrt{1+x^2}}, \quad (256)$$

which satisfies: $\mu(x) \rightarrow 1, x \gg 1$ (*Newtonian*),
 $\mu(x) \rightarrow x, x \ll 1$ (*deepMOND*).

D.8 Summary

- For a Gaussian kernel, the smoothed invariant satisfies $\bar{X} = X + \ell_{\text{coh}}^2 \nabla^2 X + \dots$.
- The scalar equation reduces to $\bar{X} = X_0$ on the non-trivial branch.
- In spherical symmetry, this yields a differential equation for $\Gamma(r)$ and $\Phi'(r)$.
- In the deep-MOND regime, the solution is $g = \sqrt{a_0 g_N}$.
- The Gaussian kernel produces the interpolation function $\mu(x) = x/\sqrt{1+x^2}$.

E Full variation of the global coherence term

In this appendix we derive the full variation of the global coherence term

$$S_{\text{coh}}^{\text{glob}} = \frac{\lambda}{2} I(t)^2, \quad I(t) = \int_{H(t)} d^4x \sqrt{-g(x)} (\bar{X}(x) - X_0), \quad (257)$$

with respect to both the scalar field ϕ and the metric $g_{\mu\nu}$. All steps are explicit and rely only on the fixed-domain prescription $\delta H(t) = 0$.

E.1 Preliminaries

The variation of $S_{\text{coh}}^{\text{glob}}$ is

$$\delta S_{\text{coh}}^{\text{glob}} = \lambda I(t) \delta I(t). \quad (258)$$

The variation of $I(t)$ is

$$\delta I(t) = \int_{H(t)} d^4x [\delta(\sqrt{-g}) (\bar{X} - X_0) + \sqrt{-g} \delta \bar{X}]. \quad (259)$$

We now compute each term separately.

E.2 Variation of the measure

The variation of the determinant is

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \quad (260)$$

Thus

$$\delta I(t)|_{\text{measure}} = -\frac{1}{2} \int_{H(t)} d^4x \sqrt{-g} (\bar{X} - X_0) g_{\mu\nu} \delta g^{\mu\nu}. \quad (261)$$

E.3 Variation of the smoothed invariant

The smoothed invariant is

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y). \quad (262)$$

Its variation is

$$\delta\bar{X}(x) = \int d^4y \left[\delta(\sqrt{-g(y)}) K(x, y) X(y) + \sqrt{-g(y)} K(x, y) \delta X(y) + \sqrt{-g(y)} X(y) \delta K(x, y) \right]. \quad (263)$$

We analyze each term.

E.3.1 Variation of the measure inside \bar{X}

Using eq:delta-sqrtg-appE,

$$\delta\bar{X}(x)|_{\text{measure}} = -\frac{1}{2} \int d^4y \sqrt{-g(y)} g_{\mu\nu}(y) K(x, y) X(y) \delta g^{\mu\nu}(y). \quad (264)$$

E.3.2 Variation of X

The kinetic invariant is

$$X(y) = \frac{1}{2} g^{\mu\nu}(y) \partial_\mu \phi \partial_\nu \phi. \quad (265)$$

Thus

$$\delta X(y) = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \delta g^{\mu\nu} + g^{\mu\nu} \partial_\mu \phi \partial_\nu \delta \phi. \quad (266)$$

The first term contributes to the metric variation; the second contributes to the scalar equation.

E.3.3 Variation of the kernel

The kernel depends on the metric through the geodesic distance:

$$K(x, y) = F\left(\frac{d^2(x, y)}{\ell_{\text{coh}}^2}\right). \quad (267)$$

Thus

$$\delta K(x, y) = F' \left(\frac{d^2}{\ell_{\text{coh}}^2} \right) \frac{1}{\ell_{\text{coh}}^2} \delta d^2(x, y). \quad (268)$$

The variation of the squared geodesic distance is

$$\delta d^2(x, y) = - \int_0^1 d\lambda \dot{\gamma}^\mu(\lambda) \dot{\gamma}^\nu(\lambda) \delta g_{\mu\nu}(\gamma(\lambda)), \quad (269)$$

where $\gamma(\lambda)$ is the geodesic connecting x and y .

This term contributes only to the metric variation.

E.4 Scalar-field variation

Collecting all terms proportional to $\delta\phi$, we obtain $\delta I(t)|_{\delta\phi} = \int_{H(t)} d^4x \sqrt{-g(x)} \int d^4y \sqrt{-g(y)} K(x, y) g^{\mu\nu}(y) \partial_\mu \phi(y) \delta\phi(y)$.

Integrating by parts in y and using the boundary condition $\delta\phi|_{\partial H(t)} = 0$,

$$\delta I(t)|_{\delta\phi} = - \int d^4y \sqrt{-g(y)} \nabla_\nu^{(y)} [\mathcal{K}(y; t) \partial^\nu \phi(y)] \delta\phi(y), \quad (270)$$

where

$$\mathcal{K}(y; t) \equiv \int_{H(t)} d^4x \sqrt{-g(x)} K(x, y). \quad (271)$$

Thus the scalar contribution to the global term is

$$\delta_\phi S_{\text{coh}}^{\text{glob}} = -\lambda I(t) \int d^4y \sqrt{-g(y)} \nabla_\nu [\mathcal{K}(y; t) \partial^\nu \phi(y)] \delta\phi(y). \quad (272)$$

This reproduces the scalar equation of motion in Section 2.

E.5 Metric variation

Collecting all terms proportional to $\delta g^{\mu\nu}$, we obtain $\delta I(t)|_{\delta g} = -\frac{1}{2} \int_{H(t)} d^4x \sqrt{-g(x)} (\bar{X} - X_0) g_{\mu\nu}(x) \delta g^{\mu\nu}(x)$

$$+ \int_{H(t)} d^4x \sqrt{-g(x)} \int d^4y \sqrt{-g(y)} K(x, y) \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \delta g^{\mu\nu}(y)$$

$$- \frac{1}{2} \int_{H(t)} d^4x \sqrt{-g(x)} \int d^4y \sqrt{-g(y)} g_{\mu\nu}(y) K(x, y) X(y) \delta g^{\mu\nu}(y)$$

$$+ \int_{H(t)} d^4x \sqrt{-g(x)} \int d^4y \sqrt{-g(y)} X(y) \delta K(x, y) \delta g^{\mu\nu}(y).$$

The last term contains the geodesic integral eq:d2-variation-appE.

The metric variation of the global term is then

$$\delta_g S_{\text{coh}}^{\text{glob}} = \lambda I(t) \delta I(t)|_{\delta g}. \quad (273)$$

The corresponding stress-energy tensor is

$$T_{\mu\nu}^{\text{coh, glob}} = - \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{coh}}^{\text{glob}}}{\delta g^{\mu\nu}}. \quad (274)$$

E.6 Summary

- The variation of the global term splits into measure, kernel, and kinetic contributions.
- The scalar variation yields $\nabla_\nu[\mathcal{K}(x; t) \partial^\nu \phi(x)] = 0$, multiplied by the global factor $\lambda I(t)$.
- The metric variation contains:
 - a local term proportional to $(\bar{X} - X_0)g_{\mu\nu}$,
 - a non-local term involving $K(x, y)$,
 - a geodesic-distance term from δK .
- These contributions define the global coherence stress–energy tensor $T_{\mu\nu}^{\text{coh. glob.}}$.

F Hamiltonian analysis

In this appendix we perform a Hamiltonian analysis of the scalar sector of the coherence theory. The goal is to determine the number of propagating degrees of freedom and to verify the absence of Ostrogradsky instabilities. The analysis is carried out in the ADM formalism and relies only on the structure of the action derived in Section 2.

F.1 ADM decomposition

We decompose the metric as

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (275)$$

where N is the lapse, N^i the shift, and h_{ij} the spatial metric.

The scalar field is decomposed as

$$\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x}), \quad (276)$$

but for the Hamiltonian analysis we keep the full field ϕ .

The kinetic invariant becomes

$$X = \frac{1}{2N^2}(\dot{\phi} - N^i \partial_i \phi)^2 - \frac{1}{2}h^{ij} \partial_i \phi \partial_j \phi. \quad (277)$$

The smoothed invariant is

$$\bar{X}(x) = \int d^3y N(y) \sqrt{h(y)} K_{\ell_{\text{coh}}}(x, y) X(y). \quad (278)$$

F.2 Structure of the scalar Lagrangian

The scalar Lagrangian density is

$$\mathcal{L}_\phi = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (279)$$

Crucially, the Lagrangian depends on $\dot{\phi}$ only through X and therefore only quadratically:

$$X = \frac{1}{2N^2}(\dot{\phi} - N^i \partial_i \phi)^2 + \dots \quad (280)$$

Thus the Lagrangian is at most quadratic in $\dot{\phi}$ and contains no higher time derivatives. This is the key structural reason why no Ostrogradsky instability can arise.

F.3 Canonical momentum

The canonical momentum conjugate to ϕ is

$$\pi_\phi \equiv \frac{\partial \mathcal{L}_\phi}{\partial \dot{\phi}}. \quad (281)$$

Using the chain rule,

$$\pi_\phi = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \frac{\partial \bar{X}}{\partial \dot{\phi}}. \quad (282)$$

From eq:Xbar-ADM-appF,

$$\frac{\partial \bar{X}(x)}{\partial \dot{\phi}(y)} = N(y) \sqrt{h(y)} K(x, y) \frac{\partial X(y)}{\partial \dot{\phi}(y)}. \quad (283)$$

And from eq:X-ADM-appF,

$$\frac{\partial X}{\partial \dot{\phi}} = \frac{1}{N^2}(\dot{\phi} - N^i \partial_i \phi). \quad (284)$$

Thus

$$\pi_\phi(x) = \frac{2}{\Lambda_{\text{coh}}^4} \int d^3y N(y) \sqrt{h(y)} K(x, y) (\bar{X}(y) - X_0) \frac{\dot{\phi}(y) - N^i(y) \partial_i \phi(y)}{N(y)^2}. \quad (285)$$

Key structural fact. The momentum is *linear* in $\dot{\phi}$.

Thus the Legendre transform is invertible and the Hamiltonian is well-defined.

F.4 Hamiltonian density

The Hamiltonian density is

$$\mathcal{H}_\phi = \pi_\phi \dot{\phi} - \mathcal{L}_\phi. \quad (286)$$

Using eq:pi-final-appF and the quadratic structure of \mathcal{L}_ϕ , we obtain

$$\mathcal{H}_\phi = \frac{\Lambda_{\text{coh}}^4}{4} \int d^3y d^3z \pi_\phi(y) \mathcal{K}^{-1}(y, z) \pi_\phi(z) + \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}, \quad (287)$$

where \mathcal{K}^{-1} is the inverse kernel operator defined by

$$\int d^3z \mathcal{K}(x, z) \mathcal{K}^{-1}(z, y) = \delta^{(3)}(x - y). \quad (288)$$

Positivity. Both terms in eq:H-final-appF are manifestly non-negative:

- the first term is a positive-definite quadratic form in π_ϕ ,
- the second term is a square.

Thus

$$\mathcal{H}_\phi \geq 0. \quad (289)$$

There is no ghost or instability in the Hamiltonian.

F.5 Degrees of freedom

The scalar field has:

- one canonical coordinate ϕ ,
- one canonical momentum π_ϕ ,
- no primary constraints,
- no secondary constraints.

Thus the scalar sector propagates exactly **one** degree of freedom.

The metric sector propagates the usual two tensor modes of GR.

Total degrees of freedom:

$$N_{\text{dof}} = 2_{\text{graviton}} + 1_{\text{scalar}} = 3. \quad (290)$$

F.6 Absence of Ostrogradsky instabilities

Ostrogradsky instabilities arise when the Lagrangian depends on higher time derivatives (e.g. $\ddot{\phi}$) in a non-degenerate way.

In this theory:

- X contains only first derivatives of ϕ ,
- \bar{X} is a linear functional of X ,
- $(\bar{X} - X_0)^2$ contains no higher derivatives,
- the kernel introduces spatial non-locality but no temporal non-locality.

Thus the Lagrangian contains no higher time derivatives, and the Hamiltonian is bounded below.

F.7 Summary

- The scalar momentum is linear in $\dot{\phi}$, so the Legendre transform is invertible.
- The Hamiltonian is positive definite: $H_\phi \geq 0$.
- The scalar sector propagates exactly one degree of freedom.
- No primary or secondary constraints arise.
- No Ostrogradsky instability is present.
- The full theory propagates three degrees of freedom: two tensor modes and one scalar mode.

G Covariant Green's function representation of the kernel

In this appendix we show that the coherence kernel $K_{\ell_{\text{coh}}}(x, y)$ can be represented as the Green's function of a covariant elliptic operator. This representation clarifies the mathematical structure of the smoothing operation, its derivative expansion, and its relation to non-local operators of the form $(1 - \ell_{\text{coh}}^2 \nabla^2)^{-1}$.

G.1 Motivation

The smoothed invariant is defined by

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y), \quad (291)$$

where $K_{\ell_{\text{coh}}}(x, y)$ is a covariant, normalized, symmetric kernel with characteristic width ℓ_{coh} .

A natural question is whether $K_{\ell_{\text{coh}}}$ can be written as the Green's function of a differential operator. This is useful because:

- it provides a covariant definition of smoothing,
- it clarifies the derivative expansion,
- it connects the theory to non-local gravity models,
- it ensures positivity and regularity of the kernel.

G.2 Covariant elliptic operator

Consider the operator

$$\mathcal{D} \equiv 1 - \ell_{\text{coh}}^2, \quad (292)$$

where $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d'Alembertian.

We define the Green's function $G_{\ell_{\text{coh}}}(x, y)$ by

$$\mathcal{D}_x G_{\ell_{\text{coh}}}(x, y) = \frac{\delta^{(4)}(x - y)}{\sqrt{-g(y)}}. \quad (293)$$

This is a covariant generalization of the Yukawa Green's function.

Properties.

- \mathcal{D} is elliptic on static slices.
- $G_{\ell_{\text{coh}}}$ is symmetric: $G_{\ell_{\text{coh}}}(x, y) = G_{\ell_{\text{coh}}}(y, x)$.
- $G_{\ell_{\text{coh}}}$ is positive and decays exponentially on scales ℓ_{coh} .

G.3 Kernel as a normalized Green's function

We define the kernel as

$$K_{\ell_{\text{coh}}}(x, y) = \frac{G_{\ell_{\text{coh}}}(x, y)}{\int d^4 z \sqrt{-g(z)} G_{\ell_{\text{coh}}}(x, z)}. \quad (294)$$

This ensures the normalization condition

$$\int d^4 y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) = 1. \quad (295)$$

Thus the smoothed invariant becomes

$$\bar{X}(x) = \frac{\int d^4 y \sqrt{-g(y)} G_{\ell_{\text{coh}}}(x, y) X(y)}{\int d^4 z \sqrt{-g(z)} G_{\ell_{\text{coh}}}(x, z)}. \quad (296)$$

G.4 Small- ℓ_{coh} expansion

Expanding the operator inverse,

$$\mathcal{D}^{-1} = (1 - \ell_{\text{coh}}^2)^{-1} = 1 + \ell_{\text{coh}}^2 + \ell_{\text{coh}}^4 + \dots, \quad (297)$$

we obtain

$$G_{\ell_{\text{coh}}}(x, y) = \frac{\delta^{(4)}(x - y)}{\sqrt{-g(y)}} + \ell_{\text{coh}}^2 \frac{\delta^{(4)}(x - y)}{\sqrt{-g(y)}} + \mathcal{O}(\ell_{\text{coh}}^4). \quad (298)$$

Substituting into eq: Xbar-Green-appG,

$$\bar{X}(x) = X(x) + \ell_{\text{coh}}^2 X(x) + \mathcal{O}(\ell_{\text{coh}}^4), \quad (299)$$

in agreement with Appendix A.

G.5 Static weak-field limit

In a static weak-field region, the operator reduces to

$$\mathcal{D} = 1 - \ell_{\text{coh}}^2 \nabla^2, \quad (300)$$

and the Green's function becomes the Yukawa kernel

$$G_{\ell_{\text{coh}}}(r) = \frac{1}{4\pi\ell_{\text{coh}}^2} \frac{e^{-r/\ell_{\text{coh}}}}{r}. \quad (301)$$

Thus the kernel is

$$K_{\ell_{\text{coh}}}(r) = \frac{\frac{e^{-r/\ell_{\text{coh}}}}{r}}{\int_0^\infty dr' 4\pi r'^2 \frac{e^{-r'/\ell_{\text{coh}}}}{r'}} = \frac{e^{-r/\ell_{\text{coh}}}}{4\pi\ell_{\text{coh}}^2 r}. \quad (302)$$

This is the static analogue of the Gaussian kernel used in Appendix D.

G.6 Non-local operator representation

Using the Green's function representation, the smoothing operation can be written compactly as

$$\bar{X} = \frac{\mathcal{D}^{-1}X}{\mathcal{D}^{-1}\mathbb{1}}. \quad (303)$$

In the limit where $\mathcal{D}^{-1}\mathbb{1} \approx \mathbb{1}$ (e.g. static or homogeneous configurations), this reduces to

$$\bar{X} \approx (1 - \ell_{\text{coh}}^2)^{-1}X. \quad (304)$$

Thus the coherence smoothing is equivalent to applying a non-local operator of Yukawa type.

G.7 Summary

- The coherence kernel can be represented as a normalized Green's function of the operator $\mathcal{D} = 1 - \ell_{\text{coh}}^2$.
- This representation is covariant, symmetric, and positive.
- The derivative expansion follows from the operator expansion $(1 - \ell_{\text{coh}}^2)^{-1}$.
- In the static weak-field limit, the kernel reduces to a Yukawa kernel.
- The smoothing operation is equivalent to applying a non-local operator $(1 - \ell_{\text{coh}}^2)^{-1}$ to X .

H Numerical implementation details

This appendix provides practical guidance for numerically solving the equations of motion derived in the main text. The coherence theory contains non-local structure through the smoothing kernel $K_{\ell_{\text{coh}}}$, and non-linear structure through the scalar equation of motion. We present stable and efficient numerical schemes for:

- computing the smoothed invariant \bar{X} ,
- evolving the FRW background,
- solving the static MOND-like equation,
- computing linear perturbations,
- handling the kernel in real and Fourier space.

All methods are designed to preserve covariance, stability, and accuracy.

H.1 Discretization of the kernel

The smoothed invariant is

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y). \quad (305)$$

For numerical work, we discretize spacetime into a lattice of points x_n with weights $w_n = \sqrt{-g(x_n)} \Delta^4x$. The kernel becomes a matrix:

$$K_{mn} \equiv K_{\ell_{\text{coh}}}(x_m, x_n). \quad (306)$$

Then

$$\bar{X}_m = \sum_n K_{mn} w_n X_n. \quad (307)$$

Normalization. To ensure numerical stability, we explicitly enforce

$$\sum_n K_{mn} w_n = 1 \quad (308)$$

for each m .

H.2 Fourier-space implementation

For static or homogeneous configurations, the kernel depends only on the geodesic distance, and the smoothing becomes a convolution:

$$\bar{X}(\mathbf{x}) = \int d^3y K(|\mathbf{x} - \mathbf{y}|) X(\mathbf{y}). \quad (309)$$

Thus

$$\bar{X}(\mathbf{k}) = K(\mathbf{k}) X(\mathbf{k}), \quad (310)$$

where $K(\mathbf{k})$ is the Fourier transform of the kernel.

For the Gaussian kernel,

$$K(\mathbf{k}) = \exp\left(-\frac{1}{2}\ell_{\text{coh}}^2 k^2\right). \quad (311)$$

This is the most efficient implementation for:

- FRW background evolution,
- linear perturbations,
- static spherical solutions.

H.3 FRW background evolution

The background equations are: $3M_{\text{Pl}}^2 H^2 = \rho_m + \rho_r + \rho_\phi$,

$$\rho_\phi = \frac{(X - X_0)^2}{\Lambda_{\text{coh}}^4},$$

$$X = \frac{1}{2}\dot{\phi}^2,$$

$$\frac{d}{dt} [a^3 \mathcal{C}(t) \dot{\phi}] = 0.$$

A stable numerical scheme is:

Step 1: Evolve ϕ . Use a symplectic integrator (e.g. leapfrog): $\dot{\phi}_{n+1/2} = \dot{\phi}_{n-1/2} - \Delta t \frac{d}{dt} \ln[a^3 \mathcal{C}(t)] \dot{\phi}_n$,
 $\phi_{n+1} = \phi_n + \Delta t \dot{\phi}_{n+1/2}$.

Step 2: Compute X and ρ_ϕ .

$$X_n = \frac{1}{2}\dot{\phi}_n^2, \quad \rho_{\phi,n} = \frac{(X_n - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (312)$$

Step 3: Update $a(t)$. Use a standard Friedmann integrator:

$$a_{n+1} = a_n + \Delta t a_n H_n. \quad (313)$$

This scheme is stable because the scalar equation is first order in $a^3 \mathcal{C} \dot{\phi}$.

H.4 Static MOND-like equation

The scalar equation reduces to

$$X(r) + \ell_{\text{coh}}^2 \nabla^2 X(r) = X_0. \quad (314)$$

A stable numerical scheme is:

Step 1: Discretize r . Use a logarithmic grid:

$$r_i = r_{\text{min}} \exp(i \Delta \ln r). \quad (315)$$

Step 2: Discretize the Laplacian.

$$\nabla^2 X_i = \frac{1}{r_i^2} \frac{d}{dr_i} \left(r_i^2 \frac{dX}{dr_i} \right) \quad (316)$$

with second-order finite differences.

Step 3: Solve the non-linear equation. Use Newton–Raphson iteration:

$$X^{(n+1)} = X^{(n)} - \frac{F[X^{(n)}]}{F'[X^{(n)}]}, \quad (317)$$

where

$$F[X] = X + \ell_{\text{coh}}^2 \nabla^2 X - X_0. \quad (318)$$

Step 4: Recover $\phi'(r)$ and $\Phi'(r)$. Use the alignment relation

$$X = \frac{1}{2} \phi'^2, \quad \phi' = \Gamma(r) \Phi'. \quad (319)$$

H.5 Linear perturbations

Because the scalar sector does not contribute at linear order,

$$\delta T_{\mu\nu}^{(\phi)} = 0, \quad (320)$$

the linear perturbation equations reduce to those of GR.

Thus standard Boltzmann codes (CLASS, CAMB) can be used with a modified background expansion $H(a)$.

Implementation.

- Replace the dark-energy density with $\rho_\phi(a)$.
- Keep all perturbation modules unchanged.
- Ensure that $\rho_\phi(a)$ and $p_\phi(a)$ satisfy $w_\phi = 1$ except at late times.

H.6 Kernel evaluation in curved space

For general spacetimes, the kernel depends on the geodesic distance $d(x, y)$.

A stable numerical scheme is:

Step 1: Compute geodesics. Use a shooting method or Runge–Kutta integration of the geodesic equation.

Step 2: Compute $d(x, y)$. Integrate the line element along the geodesic.

Step 3: Evaluate the kernel.

$$K(x, y) = \frac{1}{(2\pi\ell_{\text{coh}}^2)^2} \exp\left[-\frac{d^2(x, y)}{2\ell_{\text{coh}}^2}\right]. \quad (321)$$

Step 4: Normalize.

$$K(x, y) \rightarrow \frac{K(x, y)}{\int d^4z \sqrt{-g(z)} K(x, z)}. \quad (322)$$

H.7 Summary

- The kernel can be implemented in real space or Fourier space.
- FRW evolution is stable using a symplectic integrator.
- The static MOND-like equation is solved using Newton–Raphson iteration.
- Linear perturbations can be computed with standard Boltzmann codes.
- Kernel evaluation in curved space requires geodesic integration.

I Covariant perturbation theory of the kernel

In this appendix we develop a systematic perturbation theory for the coherence kernel $K_{\ell_{\text{coh}}}(x, y)$ under small perturbations of the metric and the scalar field. This formalism is required for computing second-order perturbations, non-linear lensing, and the response of the smoothed invariant \bar{X} to metric fluctuations.

I.1 Setup

We consider perturbations of the metric and scalar field: $g_{\mu\nu}(x) = g_{\mu\nu}^{(0)}(x) + h_{\mu\nu}(x)$, $\phi(x) = \phi_0(x) + \delta\phi(x)$, where $h_{\mu\nu}$ and $\delta\phi$ are small.

The kernel depends on the metric through the geodesic distance:

$$K_{\ell_{\text{coh}}}(x, y) = F\left(\frac{d^2(x, y)}{\ell_{\text{coh}}^2}\right), \quad (323)$$

where F is a smooth, rapidly decaying function (e.g. Gaussian or Yukawa).

We expand the kernel as

$$K(x, y) = K^{(0)}(x, y) + \delta K(x, y) + \delta^2 K(x, y) + \dots \quad (324)$$

I.2 Perturbation of the squared geodesic distance

The squared geodesic distance is

$$d^2(x, y) = \int_0^1 d\lambda g_{\mu\nu}(\gamma(\lambda)) \dot{\gamma}^\mu(\lambda) \dot{\gamma}^\nu(\lambda), \quad (325)$$

where $\gamma(\lambda)$ is the geodesic connecting x and y .

The first-order variation is

$$\delta d^2(x, y) = \int_0^1 d\lambda h_{\mu\nu}(\gamma(\lambda)) \dot{\gamma}^\mu \dot{\gamma}^\nu. \quad (326)$$

The second-order variation is $\delta^2 d^2(x, y) = \int_0^1 d\lambda [\delta\gamma^\rho \partial_\rho h_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu + h_{\mu\nu} \delta(\dot{\gamma}^\mu \dot{\gamma}^\nu)] + \int_0^1 d\lambda h_{\mu\nu} h_{\rho\sigma} \dot{\gamma}^\mu \dot{\gamma}^\nu \dot{\gamma}^\rho \dot{\gamma}^\sigma + \dots$, where $\delta\gamma^\mu$ is the first-order perturbation of the geodesic.

I.3 Perturbation of the kernel

Using the chain rule,

$$\delta K(x, y) = F'\left(\frac{d_0^2}{\ell_{\text{coh}}^2}\right) \frac{\delta d^2(x, y)}{\ell_{\text{coh}}^2}, \quad (327)$$

and

$$\delta^2 K(x, y) = F''\left(\frac{d_0^2}{\ell_{\text{coh}}^2}\right) \frac{(\delta d^2)^2}{\ell_{\text{coh}}^4} + F'\left(\frac{d_0^2}{\ell_{\text{coh}}^2}\right) \frac{\delta^2 d^2}{\ell_{\text{coh}}^2}. \quad (328)$$

These expressions are exact and covariant.

I.4 Perturbation of the smoothed invariant

The smoothed invariant is

$$\bar{X}(x) = \int d^4 y \sqrt{-g(y)} K(x, y) X(y). \quad (329)$$

We expand

$$\bar{X} = \bar{X}^{(0)} + \delta \bar{X} + \delta^2 \bar{X} + \dots \quad (330)$$

I.4.1 First-order variation

$$\delta\bar{X}(x) = \int d^4y \sqrt{-g_0(y)} [\delta K(x, y) X_0(y) + K^{(0)}(x, y) \delta X(y)] + \int d^4y \delta(\sqrt{-g(y)}) K^{(0)}(x, y) X_0(y).$$

Using

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} h, \quad h \equiv g^{\mu\nu} h_{\mu\nu}, \quad (331)$$

and

$$\delta X = g^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \delta\phi - \frac{1}{2} h^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \phi_0, \quad (332)$$

we obtain the full first-order expression.

I.4.2 Second-order variation

The second-order variation is $\delta^2\bar{X}(x) = \int d^4y \sqrt{-g_0(y)} [\delta^2 K(x, y) X_0(y) + 2\delta K(x, y) \delta X(y) + K^{(0)}(x, y) \delta^2 X(y)] + \int d^4y \delta(\sqrt{-g}) \delta K X_0 + \int d^4y \delta^2(\sqrt{-g}) K^{(0)} X_0$.

This expression is required for non-linear lensing and second-order perturbation theory.

I.5 Perturbation of the normalization

The kernel satisfies

$$\int d^4y \sqrt{-g(y)} K(x, y) = 1. \quad (333)$$

Expanding,

$$\delta K + \delta\sqrt{-g} K^{(0)} = 0, \quad (334)$$

and

$$\delta^2 K + 2\delta\sqrt{-g} \delta K + \delta^2\sqrt{-g} K^{(0)} = 0. \quad (335)$$

These relations ensure that the kernel remains normalized at each perturbative order.

I.6 Summary

- The kernel perturbations are determined by perturbations of the squared geodesic distance.
- The first- and second-order variations of K are given by $\delta K = F' \delta d^2 / \ell_{\text{coh}}^2$, $\delta^2 K = F'' (\delta d^2)^2 / \ell_{\text{coh}}^4 + F' \delta^2 d^2 / \ell_{\text{coh}}^2$.
- The smoothed invariant satisfies $\delta\bar{X} = \int (\delta K X_0 + K^{(0)} \delta X)$, $\delta^2\bar{X} = \int (\delta^2 K X_0 + 2\delta K \delta X + K^{(0)} \delta^2 X)$.
- Kernel normalization imposes constraints on δK and $\delta^2 K$.
- These results are required for non-linear lensing and second-order cosmological perturbation theory.

J Analytic solutions in special geometries

This appendix collects analytic solutions of the scalar equation of motion and the modified Poisson equation in geometries of high symmetry. These solutions are useful for understanding the structure of the coherence theory in astrophysical and cosmological settings.

The scalar equation in the static limit is

$$X + \ell_{\text{coh}}^2 \nabla^2 X = X_0, \quad (336)$$

where $X = \frac{1}{2} |\nabla \phi|^2$ and the gradients of ϕ and Φ are aligned:

$$\nabla \phi = \Gamma(\mathbf{x}) \nabla \Phi. \quad (337)$$

The modified Poisson equation is

$$\nabla \cdot \left[\mu \left(\frac{|\nabla \Phi|}{a_0} \right) \nabla \Phi \right] = 4\pi G \rho_b, \quad (338)$$

with the interpolation function determined by the solution of eq:scalar-branch-appJ.

We now solve eq:scalar-branch-appJ in several geometries.

J.1 Homogeneous FRW spacetime

The FRW metric is

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2. \quad (339)$$

Homogeneity implies $X = X(t)$ and $\nabla^2 X = 0$. Thus eq:scalar-branch-appJ reduces to

$$X(t) = X_0. \quad (340)$$

The non-trivial kinetic branch is obtained from the full scalar equation:

$$X(t) = \frac{C^2}{2a(t)^6}. \quad (341)$$

This reproduces the exact FRW solution of Section 4.

J.2 Spherical symmetry

For a static, spherically symmetric configuration,

$$X = X(r), \quad \nabla^2 X = X'' + \frac{2}{r} X'. \quad (342)$$

The scalar equation becomes

$$X(r) + \ell_{\text{coh}}^2 \left(X'' + \frac{2}{r} X' \right) = X_0. \quad (343)$$

J.2.1 General solution

The homogeneous equation

$$X + \ell_{\text{coh}}^2 \left(X'' + \frac{2}{r} X' \right) = 0 \quad (344)$$

has solutions

$$X_h(r) = \frac{A}{r} e^{-r/\ell_{\text{coh}}} + \frac{B}{r} e^{+r/\ell_{\text{coh}}}. \quad (345)$$

Regularity at infinity requires $B = 0$.

A particular solution of eq:spherical-X-eq-appJ is

$$X_p = X_0. \quad (346)$$

Thus the full solution is

$$X(r) = X_0 + \frac{A}{r} e^{-r/\ell_{\text{coh}}}. \quad (347)$$

Deep-MOND regime. When $X \ll X_0$, the exponential term dominates:

$$X(r) \approx \frac{A}{r} e^{-r/\ell_{\text{coh}}}. \quad (348)$$

Newtonian regime. When $r \ll \ell_{\text{coh}}$,

$$X(r) \approx X_0 + \frac{A}{r}. \quad (349)$$

J.3 Planar symmetry (1D)

For a planar mass distribution (galactic disk approximation),

$$X = X(z), \quad \nabla^2 X = X''(z). \quad (350)$$

The scalar equation becomes

$$X(z) + \ell_{\text{coh}}^2 X''(z) = X_0. \quad (351)$$

J.3.1 General solution

The homogeneous equation

$$X + \ell_{\text{coh}}^2 X'' = 0 \quad (352)$$

has solutions

$$X_h(z) = C_1 e^{-z/\ell_{\text{coh}}} + C_2 e^{+z/\ell_{\text{coh}}}. \quad (353)$$

For a symmetric disk, $X(z) = X(-z)$, so $C_1 = C_2$.

A particular solution is X_0 .

Thus

$$X(z) = X_0 + C \cosh\left(\frac{z}{\ell_{\text{coh}}}\right). \quad (354)$$

Deep-MOND regime. Near the midplane ($z \ll \ell_{\text{coh}}$),

$$X(z) \approx X_0 + C \left(1 + \frac{z^2}{2\ell_{\text{coh}}^2} \right). \quad (355)$$

Asymptotic regime. For $|z| \gg \ell_{\text{coh}}$,

$$X(z) \sim C e^{|z|/\ell_{\text{coh}}}. \quad (356)$$

J.4 Cylindrical symmetry (2D)

For a thin rotating disk,

$$X = X(R), \quad \nabla^2 X = X'' + \frac{1}{R} X'. \quad (357)$$

The scalar equation becomes

$$X(R) + \ell_{\text{coh}}^2 \left(X'' + \frac{1}{R} X' \right) = X_0. \quad (358)$$

J.4.1 General solution

The homogeneous equation

$$X + \ell_{\text{coh}}^2 \left(X'' + \frac{1}{R} X' \right) = 0 \quad (359)$$

has Bessel-function solutions:

$$X_h(R) = A K_0(R/\ell_{\text{coh}}) + B I_0(R/\ell_{\text{coh}}), \quad (360)$$

where I_0 and K_0 are modified Bessel functions.

Regularity at infinity requires $B = 0$.

Thus

$$X(R) = X_0 + A K_0(R/\ell_{\text{coh}}). \quad (361)$$

Deep-MOND regime. For $R \gg \ell_{\text{coh}}$,

$$X(R) \sim A \sqrt{\frac{\pi \ell_{\text{coh}}}{2R}} e^{-R/\ell_{\text{coh}}}. \quad (362)$$

Newtonian regime. For $R \ll \ell_{\text{coh}}$,

$$X(R) \approx X_0 - A \ln(R/\ell_{\text{coh}}). \quad (363)$$

J.5 Asymptotic matching and rotation curves

In cylindrical symmetry, the gravitational acceleration is

$$g(R) = \Phi'(R). \quad (364)$$

Using the alignment relation,

$$X = \frac{1}{2}\Gamma^2 g^2, \quad (365)$$

and the solution eq:X-cylindrical-appJ, we obtain

$$g(R) = \frac{\sqrt{2}}{\Gamma(R)} \sqrt{X_0 + AK_0(R/\ell_{\text{coh}})}. \quad (366)$$

In the deep-MOND regime ($R \gg \ell_{\text{coh}}$),

$$g(R) \sim \frac{\sqrt{2A}}{\Gamma(R)} \left(\frac{\pi\ell_{\text{coh}}}{2R} \right)^{1/4} e^{-R/(2\ell_{\text{coh}})}. \quad (367)$$

In the Newtonian regime ($R \ll \ell_{\text{coh}}$),

$$g(R) \approx \frac{\sqrt{2}}{\Gamma(R)} \sqrt{X_0 - A \ln(R/\ell_{\text{coh}})}. \quad (368)$$

J.6 Summary

- The scalar equation admits analytic solutions in FRW, spherical, planar, and cylindrical geometries.
- In spherical symmetry, $X(r) = X_0 + Ae^{-r/\ell_{\text{coh}}}/r$.
- In planar symmetry, $X(z) = X_0 + C \cosh(z/\ell_{\text{coh}})$.
- In cylindrical symmetry, $X(R) = X_0 + AK_0(R/\ell_{\text{coh}})$.
- These solutions determine the MOND-like behavior of the gravitational field in different astrophysical settings.

K Gauge-invariant formulation

This appendix presents the gauge-invariant formulation of linear perturbations in the coherence theory. We show explicitly that the scalar sector does not contribute to the gauge-invariant Einstein equations at first order, and that the only dynamical gauge-invariant degrees of freedom are the standard GR metric perturbations and matter perturbations.

K.1 Perturbation variables

We perturb the metric as

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}, \quad (369)$$

and the scalar field as

$$\phi = \phi_0(t) + \delta\phi(t, \mathbf{x}). \quad (370)$$

In scalar–vector–tensor (SVT) decomposition: $h_{00} = -2A$,
 $h_{0i} = \partial_i B + S_i$,
 $h_{ij} = 2H_L \delta_{ij} + 2(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2) H_T + \partial_i F_j + \partial_j F_i + h_{ij}^{\text{TT}}$, with

- S_i transverse: $\partial_i S_i = 0$,
- F_i transverse: $\partial_i F_i = 0$,
- h_{ij}^{TT} transverse and traceless.

K.2 Gauge transformations

Under an infinitesimal diffeomorphism

$$x^\mu \rightarrow x^\mu + \xi^\mu, \quad \xi^\mu = (\xi^0, \partial^i \xi + \xi_T^i), \quad (371)$$

the perturbations transform as: $A \rightarrow A - \dot{\xi}^0 - H\xi^0$,
 $B \rightarrow B + \xi^0 - \dot{\xi}$,
 $H_L \rightarrow H_L - H\xi^0$,
 $H_T \rightarrow H_T - \xi$,
 $\delta\phi \rightarrow \delta\phi - \dot{\phi}_0 \xi^0$.

K.3 Gauge-invariant combinations

The standard Bardeen potentials are: $\Phi_{\text{GI}} = A - \frac{d}{dt} \left[a^2 \left(\dot{H}_T - B/a \right) \right]$,
 $\Psi_{\text{GI}} = H_L + H \left(a^2 \left(\dot{H}_T - B/a \right) \right)$.

The gauge-invariant scalar-field perturbation is

$$\delta\phi_{\text{GI}} = \delta\phi - \dot{\phi}_0 \left(a^2 \left(\dot{H}_T - B/a \right) \right). \quad (372)$$

K.4 Gauge-invariant expansion of the kinetic invariant

The kinetic invariant is

$$X = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (373)$$

Expanding to first order:

$$\delta X = \dot{\phi}_0 \delta\dot{\phi} - \dot{\phi}_0^2 A. \quad (374)$$

The gauge-invariant combination is

$$\delta X_{\text{GI}} = \dot{\phi}_0 \delta\dot{\phi}_{\text{GI}} - \dot{\phi}_0^2 \Phi_{\text{GI}}. \quad (375)$$

K.5 Gauge-invariant expansion of the smoothed invariant

The smoothed invariant is

$$\bar{X} = X + \ell_{\text{coh}}^2 X + \mathcal{O}(\ell_{\text{coh}}^4). \quad (376)$$

Thus

$$\delta\bar{X}_{\text{GI}} = \delta X_{\text{GI}} + \ell_{\text{coh}}^2 \delta X_{\text{GI}}. \quad (377)$$

K.6 Gauge-invariant expansion of the coherence Lagrangian

The coherence Lagrangian is

$$\mathcal{L}_{\text{coh}} = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (378)$$

Expanding:

$$\delta\mathcal{L}_{\text{coh}} = \frac{2(\bar{X}_0 - X_0)}{\Lambda_{\text{coh}}^4} \delta\bar{X}_{\text{GI}}. \quad (379)$$

But on the FRW background,

$$\bar{X}_0 = X_0, \quad (380)$$

so

$$\delta\mathcal{L}_{\text{coh}} = 0. \quad (381)$$

Thus the scalar sector contributes ****no gauge-invariant stress-energy at first order****.

K.7 Gauge-invariant Einstein equations

The gauge-invariant Einstein equations are: $\nabla^2\Psi_{\text{GI}} - 3H(\dot{\Psi}_{\text{GI}} + H\Phi_{\text{GI}}) = 4\pi G\delta\rho_{\text{GI}}$,
 $\dot{\Psi}_{\text{GI}} + H\Phi_{\text{GI}} = 4\pi G(\rho + p)v_{\text{GI}}$,
 $\Phi_{\text{GI}} - \Psi_{\text{GI}} = 8\pi G\pi_{\text{GI}}$, where $\delta\rho_{\text{GI}}$, v_{GI} , and π_{GI} are the gauge-invariant matter perturbations.

Because of eq:deltaLcoh-zero-appK,

$$\delta T_{\mu\nu}^{(\phi)} = 0 \quad \text{at first order}. \quad (382)$$

Thus the gauge-invariant Einstein equations are ****identical to GR****.

K.8 Gauge-invariant scalar-field equation

The scalar equation of motion is

$$\nabla_{\mu} [\mathcal{F} \partial^{\mu} \phi] = 0. \quad (383)$$

At first order,

$$\delta(\nabla_\mu[\mathcal{F}\partial^\mu\phi]) = 0. \quad (384)$$

But on the FRW background,

$$\mathcal{F}_0 = 0, \quad (385)$$

so the first-order scalar equation is identically satisfied:

$$\delta\phi_{\text{GI}} \text{ is non-dynamical at first order.} \quad (386)$$

K.9 Summary

- Gauge-invariant perturbation theory is fully consistent in the coherence framework.
- The gauge-invariant scalar perturbation $\delta\phi_{\text{GI}}$ does not appear in the Einstein equations at first order.
- The gauge-invariant smoothed invariant satisfies $\delta\bar{X}_{\text{GI}} = 0$ at first order.
- The gauge-invariant Einstein equations reduce exactly to those of GR.
- The only propagating gauge-invariant degrees of freedom at linear order are the standard GR metric and matter perturbations.

L Energy conditions and positivity

In this appendix we analyze the energy conditions satisfied by the coherence stress-energy tensor. We show that the scalar sector obeys the null energy condition (NEC), weak energy condition (WEC), and dominant energy condition (DEC) on all physically relevant branches of the theory. We also show that the Hamiltonian is positive definite, consistent with Appendix F.

L.1 Stress-energy tensor

The coherence Lagrangian is

$$\mathcal{L}_{\text{coh}} = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (387)$$

The stress-energy tensor is

$$T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}} \frac{\delta\mathcal{S}_{\text{coh}}}{\delta g^{\mu\nu}} = -2 \frac{\partial\mathcal{L}_{\text{coh}}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\text{coh}}. \quad (388)$$

Using

$$\frac{\partial\bar{X}}{\partial g^{\mu\nu}} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi, \quad (389)$$

and the linearity of \bar{X} in X , we obtain

$$T_{\mu\nu}^{(\phi)} = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \left(\frac{\partial\bar{X}}{\partial g^{\mu\nu}} \right) + g_{\mu\nu} \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (390)$$

L.2 Null energy condition (NEC)

The NEC requires

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \quad \text{for all null vectors } k^\mu. \quad (391)$$

Using eq:Tmunu-expanded-appL,

$$T_{\mu\nu}k^\mu k^\nu = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \left(k^\mu k^\nu \frac{\partial \bar{X}}{\partial g^{\mu\nu}} \right). \quad (392)$$

Since

$$\frac{\partial \bar{X}}{\partial g^{\mu\nu}} = -\frac{1}{2} \int d^4y \sqrt{-g(y)} K(x, y) \partial_\mu \phi(y) \partial_\nu \phi(y), \quad (393)$$

we obtain

$$T_{\mu\nu}k^\mu k^\nu = \frac{(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \int d^4y \sqrt{-g(y)} K(x, y) (k^\mu \partial_\mu \phi)^2. \quad (394)$$

The integrand is non-negative because:

- $K(x, y) \geq 0$,
- $(k^\mu \partial_\mu \phi)^2 \geq 0$.

Thus the NEC is satisfied whenever

$$\bar{X} - X_0 \geq 0. \quad (395)$$

This holds on:

- the FRW kinetic branch ($X \geq X_0$ at early times),
- the MOND-like static branch ($\bar{X} = X_0$),
- the Solar System high-coherence branch ($\bar{X} \approx X_0$).

L.3 Weak energy condition (WEC)

The WEC requires

$$T_{\mu\nu}u^\mu u^\nu \geq 0 \quad \text{for all timelike } u^\mu. \quad (396)$$

Using eq:Tmunu-expanded-appL, $T_{\mu\nu}u^\mu u^\nu = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \left(u^\mu u^\nu \frac{\partial \bar{X}}{\partial g^{\mu\nu}} \right) + \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}$.

The second term is manifestly non-negative.

The first term is

$$u^\mu u^\nu \frac{\partial \bar{X}}{\partial g^{\mu\nu}} = -\frac{1}{2} \int d^4y \sqrt{-g(y)} K(x, y) (u^\mu \partial_\mu \phi)^2 \leq 0. \quad (397)$$

Thus

$$T_{\mu\nu}u^\mu u^\nu = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4} - \frac{(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \int K (u^\mu \partial_\mu \phi)^2. \quad (398)$$

This is non-negative whenever

$$\bar{X} - X_0 \geq 0. \quad (399)$$

Thus the WEC is satisfied on all physical branches.

L.4 Dominant energy condition (DEC)

The DEC requires:

$$T_{\mu\nu}u^\nu \text{ is non-spacelike.} \quad (400)$$

Using the structure of eq: Tmunu-expanded-appL, the energy flux is

$$T_{\mu\nu}u^\nu = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \left(\frac{\partial \bar{X}}{\partial g^{\mu\nu}} u^\nu \right) + u_\mu \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (401)$$

The second term is timelike.

The first term is proportional to

$$\partial_\mu \phi (u^\nu \partial_\nu \phi), \quad (402)$$

which is always causal.

Thus the DEC is satisfied whenever $\bar{X} - X_0 \geq 0$.

L.5 Strong energy condition (SEC)

The SEC requires

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) u^\mu u^\nu \geq 0. \quad (403)$$

Using the explicit form of $T_{\mu\nu}$, we find

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) u^\mu u^\nu = -\frac{(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \int K (u^\mu \partial_\mu \phi)^2. \quad (404)$$

Thus the SEC is ****violated**** whenever $\bar{X} > X_0$.

This is expected and physically desirable:

- SEC violation is required for cosmic acceleration,
- the scalar sector behaves like dark energy at late times.

L.6 Positivity of the Hamiltonian

Appendix F shows that the Hamiltonian density is

$$\mathcal{H}_\phi = \frac{\Lambda_{\text{coh}}^4}{4} \int \pi_\phi \mathcal{K}^{-1} \pi_\phi + \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (405)$$

Both terms are manifestly non-negative:

- \mathcal{K}^{-1} is a positive operator,
- $(\bar{X} - X_0)^2 \geq 0$.

Thus

$$\mathcal{H}_\phi \geq 0. \quad (406)$$

L.7 Summary

- NEC, WEC, and DEC are satisfied whenever $\bar{X} - X_0 \geq 0$.
- This condition holds on all physically relevant branches: FRW, MOND-like, and Solar System.
- The SEC is violated, as required for cosmic acceleration.
- The Hamiltonian is positive definite.
- The scalar sector is physically consistent and free of pathologies.

M Covariant conservation laws

In this appendix we derive the covariant conservation laws satisfied by the stress–energy tensor of the coherence sector. We show that the full stress–energy tensor is covariantly conserved,

$$\nabla^\mu T_{\mu\nu}^{\text{tot}} = 0, \quad (407)$$

and that this conservation law follows directly from diffeomorphism invariance of the action, even in the presence of the non-local smoothing kernel $K_{\ell_{\text{coh}}}(x, y)$ and the global functional $I(t)$.

M.1 Diffeomorphism invariance

Under an infinitesimal diffeomorphism generated by a vector field $\xi^\mu(x)$, the fields transform as: $\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$, $\delta_\xi \phi = \xi^\mu \partial_\mu \phi$.

The total action is invariant:

$$\delta_\xi S_{\text{tot}} = 0. \quad (408)$$

We now compute this variation explicitly.

M.2 Variation of the action

The variation of the action under a diffeomorphism is

$$\delta_\xi S_{\text{tot}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} \delta_\xi g_{\mu\nu} + \mathcal{E}_\phi \delta_\xi \phi \right], \quad (409)$$

where

$$\mathcal{E}_\phi \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \phi} \quad (410)$$

is the scalar equation of motion.

Substituting the variations: $\delta_\xi S_{\text{tot}} = \int d^4x \sqrt{-g} [T^{\mu\nu} \nabla_\mu \xi_\nu + \mathcal{E}_\phi \xi^\mu \partial_\mu \phi]$.

Integrating the first term by parts: $\int d^4x \sqrt{-g} T^{\mu\nu} \nabla_\mu \xi_\nu = - \int d^4x \sqrt{-g} (\nabla_\mu T^{\mu\nu}) \xi_\nu$, where boundary terms vanish for compact support of ξ^μ .

Thus

$$\delta_\xi S_{\text{tot}} = - \int d^4x \sqrt{-g} [(\nabla_\mu T^{\mu\nu})\xi_\nu - \mathcal{E}_\phi \partial^\nu \phi \xi_\nu]. \quad (411)$$

Since ξ_ν is arbitrary and eq:diff-invariance-appM requires $\delta_\xi S_{\text{tot}} = 0$, we obtain the Noether identity:

$$\nabla_\mu T^{\mu\nu} = \mathcal{E}_\phi \partial^\nu \phi. \quad (412)$$

M.3 On-shell conservation

When the scalar equation of motion is satisfied,

$$\mathcal{E}_\phi = 0, \quad (413)$$

the Noether identity reduces to

$$\nabla_\mu T^{\mu\nu} = 0. \quad (414)$$

Thus the total stress–energy tensor is covariantly conserved.

M.4 Local and global contributions

The total stress–energy tensor splits into:

$$T_{\mu\nu}^{\text{tot}} = T_{\mu\nu}^{\text{EH}} + T_{\mu\nu}^{\text{coh,loc}} + T_{\mu\nu}^{\text{coh,glob}} + T_{\mu\nu}^{\text{matter}}. \quad (415)$$

Each term satisfies a separate conservation identity:

Einstein–Hilbert term.

$$\nabla_\mu T_{\text{EH}}^{\mu\nu} = 0 \quad (\text{Bianchi identity}). \quad (416)$$

Matter sector.

$$\nabla_\mu T_{\text{matter}}^{\mu\nu} = 0 \quad (\text{minimal coupling}). \quad (417)$$

Local coherence term. Using the scalar equation of motion,

$$\nabla_\mu T_{\text{coh,loc}}^{\mu\nu} = \mathcal{E}_\phi^{\text{loc}} \partial^\nu \phi. \quad (418)$$

Global coherence term. The global term contributes

$$\nabla_\mu T_{\text{coh,glob}}^{\mu\nu} = \mathcal{E}_\phi^{\text{glob}} \partial^\nu \phi. \quad (419)$$

Summing all contributions:

$$\nabla_\mu T_{\text{tot}}^{\mu\nu} = (\mathcal{E}_\phi^{\text{loc}} + \mathcal{E}_\phi^{\text{glob}}) \partial^\nu \phi. \quad (420)$$

But the full scalar equation of motion is

$$\mathcal{E}_\phi^{\text{loc}} + \mathcal{E}_\phi^{\text{glob}} = 0. \quad (421)$$

Thus

$$\nabla_\mu T_{\text{tot}}^{\mu\nu} = 0. \quad (422)$$

M.5 Conservation of the smoothed invariant

The smoothed invariant satisfies

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K(x, y) X(y). \quad (423)$$

Differentiating: $\nabla_\mu \bar{X}(x) = \int d^4y \sqrt{-g(y)} [(\nabla_\mu K(x, y))X(y) + K(x, y)\nabla_\mu X(y)]$.

Using the symmetry and normalization of the kernel, and the fact that $K(x, y)$ depends only on the geodesic distance, one finds:

$$\nabla_\mu \bar{X}(x) = \overline{\nabla_\mu X}(x), \quad (424)$$

i.e. smoothing commutes with covariant differentiation.

This identity is essential for the conservation of the coherence stress–energy tensor.

M.6 Summary

- Diffeomorphism invariance implies the Noether identity

$$\nabla_\mu T^{\mu\nu} = \mathcal{E}_\phi \partial^\nu \phi.$$

- On-shell, the total stress–energy tensor is covariantly conserved.
- Local and global coherence terms each satisfy a conservation identity proportional to the scalar equation of motion.
- The smoothed invariant satisfies $\nabla_\mu \bar{X} = \overline{\nabla_\mu X}$, ensuring consistency of the non-local kernel with conservation laws.
- The theory is fully consistent with the geometric structure of GR.

N Relation to the Effective Field Theory of Dark Energy

In this appendix we compare the coherence framework to the Effective Field Theory of Dark Energy (EFT-DE). We show that the coherence theory lies outside the standard EFT-DE classification because its non-local kernel structure cannot be captured by any finite set of operators in the EFT-DE expansion. Nevertheless, the theory reduces to GR at linear order, corresponding to a very special point in EFT-DE parameter space.

N.1 EFT-DE in unitary gauge

In unitary gauge, the scalar field is taken to be a function of time only:

$$\phi(t, \mathbf{x}) = \phi_0(t), \quad (425)$$

and perturbations are encoded in the metric.

$$\begin{aligned} \text{The EFT-DE action takes the schematic form } S_{\text{EFT}} = \int d^4x \sqrt{-g} \left[\frac{M^2(t)}{2} R - \right. \\ \left. \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}(\delta K)^2 - \frac{\bar{M}_3^2(t)}{2}\delta K_{ij}\delta K^{ij} + \dots \right]. \end{aligned}$$

The coefficients

$$M^2(t), \Lambda(t), c(t), M_2^4(t), \bar{M}_1^3(t), \bar{M}_2^2(t), \bar{M}_3^2(t), \dots$$

encode the dynamics of dark energy and modified gravity.

N.2 Mapping the coherence theory to EFT-DE

The coherence action is

$$S_{\text{coh}} = \int d^4x \sqrt{-g} \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4} + \frac{\lambda}{2} I(t)^2. \quad (426)$$

In unitary gauge, $X = \frac{1}{2}\dot{\phi}_0^2$ is a function of time only, and

$$\delta X = 0, \quad \delta \bar{X} = 0, \quad (427)$$

at first order.

Thus the coherence Lagrangian contributes only at second order:

$$\delta^{(1)} \mathcal{L}_{\text{coh}} = 0. \quad (428)$$

This implies the EFT-DE coefficients satisfy: $M^2(t) = M_{\text{Pl}}^2$,
 $c(t) = 0$,
 $M_2^4(t) = 0$,
 $\bar{M}_1^3(t) = 0$,
 $\bar{M}_2^2(t) = 0$,
 $\bar{M}_3^2(t) = 0$.

Thus the coherence theory corresponds to the ****GR point**** in EFT-DE parameter space:

$$\alpha_K = \alpha_B = \alpha_M = \alpha_T = 0. \quad (429)$$

N.3 Why the EFT-DE expansion fails to capture the theory

Although the linear EFT-DE parameters vanish, the coherence theory is ***not*** equivalent to GR. The reason is that the coherence Lagrangian contains non-local structure:

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y), \quad (430)$$

which cannot be represented by any finite set of local operators in the EFT-DE expansion.

The operator $(1 - \ell_{\text{coh}}^2)^{-1}$ appearing in Appendix G generates an ****infinite tower of higher-derivative operators****:

$$\bar{X} = X + \ell_{\text{coh}}^2 X + \ell_{\text{coh}}^4 X + \dots \quad (431)$$

Thus the coherence theory lies outside:

- Horndeski,
- beyond-Horndeski,
- DHOST,
- GLPV,
- standard EFT-DE.

It belongs to a distinct class:

non-local, kernel-smearing, derivative-coherent scalar–tensor theories.

N.4 Implications for linear cosmology

Because the EFT-DE parameters vanish,

$$\Phi = \Psi, \quad c_s^2 = 1, \quad \delta T_{\mu\nu}^{(\phi)} = 0, \quad (432)$$

the linear cosmological perturbations are identical to GR.

This explains:

- why CMB anisotropies are unchanged,
- why BAO are unchanged,
- why linear lensing is unchanged,
- why the growth equation is unchanged.

The only modification is through the background expansion $H(a)$.

N.5 Implications for non-linear dynamics

The EFT-DE framework cannot describe:

- the MOND-like branch,
- the non-linear scalar equation,
- the kernel-induced non-locality,
- the non-linear lensing contributions.

These effects arise only at ****second order and beyond****, outside the EFT-DE linear regime.

N.6 Summary

- The coherence theory corresponds to the GR point in EFT-DE at linear order.
- All EFT-DE parameters vanish:

$$\alpha_K = \alpha_B = \alpha_M = \alpha_T = 0.$$

- The theory lies outside the EFT-DE classification because of its non-local kernel structure.
- Linear cosmology is identical to GR.
- Non-linear dynamics (MOND-like behavior, non-linear lensing) arise from the non-local structure and cannot be captured by EFT-DE.

O Quantum corrections and radiative stability

In this appendix we analyze the radiative stability of the coherence theory. We show that the structure of the action is technically natural: quantum corrections do not generate dangerous operators, the hierarchy $\Lambda_{\text{coh}}^2 \sim M_{\text{Pl}} H_0$ is stable, and the MOND acceleration scale a_0 is protected by symmetry and dimensional analysis.

O.1 Structure of the scalar sector

The scalar Lagrangian is

$$\mathcal{L}_{\text{coh}} = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}, \quad (433)$$

where

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y), \quad X = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (434)$$

The key structural features are:

- the action depends only on first derivatives of ϕ ,
- the dependence is quadratic in $(\bar{X} - X_0)$,
- the kernel introduces non-locality on a fixed physical scale ℓ_{coh} ,
- the theory contains no potential $V(\phi)$ and no mass term.

These features strongly constrain quantum corrections.

O.2 Absence of a local potential

Because the action depends only on derivatives of ϕ , the theory enjoys a shift symmetry:

$$\phi \rightarrow \phi + c, \quad (435)$$

for constant c .

This symmetry forbids:

- a mass term $m^2\phi^2$,
- a potential $V(\phi)$,
- any operator depending on ϕ without derivatives.

Thus quantum corrections cannot generate a potential.

O.3 Radiative stability of the kinetic structure

The operator $(\bar{X} - X_0)^2$ expands as

$$(\bar{X} - X_0)^2 = (X - X_0)^2 + 2(X - X_0)\ell_{\text{coh}}^2 X + \mathcal{O}(\ell_{\text{coh}}^4). \quad (436)$$

Quantum corrections can renormalize the coefficients of these operators, but cannot generate new structures because:

- the shift symmetry forbids non-derivative operators,
- the kernel structure is linear in X ,
- the action contains no higher time derivatives.

Thus the most general radiative correction is

$$\Delta\mathcal{L} = c_1(X - X_0)^2 + c_2(X - X_0)X + c_3(X)^2 + \dots, \quad (437)$$

with coefficients suppressed by powers of Λ_{coh} .

O.4 Non-locality and UV sensitivity

The kernel is non-local on scales ℓ_{coh} , but this non-locality is soft:

$$K_{\ell_{\text{coh}}}(x, y) = \frac{1}{(2\pi\ell_{\text{coh}}^2)^2} \exp\left[-\frac{d^2(x, y)}{2\ell_{\text{coh}}^2}\right]. \quad (438)$$

This is equivalent to the operator

$$\bar{X} = (1 - \ell_{\text{coh}}^2)^{-1}X, \quad (439)$$

which is analytic in .

Thus the theory contains no hard non-locality and no acausal behavior.

Quantum corrections cannot generate non-analytic operators such as $\sqrt{}$ or \log because:

- the theory is local at scales $\ll \ell_{\text{coh}}$,
- the kernel is exponentially suppressed at large momenta,
- loop integrals are UV-finite due to the Gaussian cutoff.

O.5 Radiative stability of the MOND scale

The MOND acceleration scale is

$$a_0 \sim \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}. \quad (440)$$

Quantum corrections could in principle renormalize Λ_{coh} or ℓ_{coh} , but:

(i) Λ_{coh} **is protected by dimensional analysis.** The only scale in the scalar sector is Λ_{coh} , so loop corrections must be proportional to it:

$$\delta\Lambda_{\text{coh}}^4 \sim \Lambda_{\text{coh}}^4. \quad (441)$$

Thus the hierarchy

$$\Lambda_{\text{coh}}^2 \sim M_{\text{Pl}} H_0$$

is technically natural.

(ii) ℓ_{coh} **is protected by the kernel.** The kernel introduces a fixed physical length scale. Quantum corrections cannot renormalize it because:

- the kernel is not dynamical,
- it is defined at the level of the action,
- it is protected by diffeomorphism invariance.

Thus

$$\delta\ell_{\text{coh}} = 0. \quad (442)$$

(iii) **Therefore a_0 is radiatively stable.**

O.6 Absence of dangerous operators

Quantum corrections cannot generate:

- higher time derivatives (protected by the structure of X),
- non-local operators beyond the kernel (protected by analyticity),
- a potential (protected by shift symmetry),
- a mass term (same reason),

- couplings to matter (protected by minimal coupling).

Thus the theory is free of:

- Ostrogradsky ghosts,
- tachyonic instabilities,
- strong-coupling pathologies,
- radiative destabilization.

O.7 Summary

- The scalar sector enjoys a shift symmetry that forbids a potential and protects the kinetic structure.
- The kernel introduces soft non-locality that is UV-finite and radiatively stable.
- The MOND acceleration scale a_0 is technically natural.
- No dangerous operators are generated by loops.
- The theory is radiatively stable and free of quantum pathologies.

P Causality and hyperbolicity

This appendix analyzes the causal structure and hyperbolicity of the coherence theory. We show that the scalar sector propagates strictly within the gravitational light cone, that the kernel does not introduce acausal behavior, and that the full system of equations is well-posed.

P.1 Principal part of the scalar equation

The scalar equation of motion is

$$\nabla_\mu [\mathcal{F}(x) \partial^\mu \phi(x)] = 0, \quad (443)$$

where

$$\mathcal{F}(x) = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4}. \quad (444)$$

The principal (highest-derivative) part is

$$\mathcal{F}(x) \phi. \quad (445)$$

Thus the characteristic surfaces satisfy

$$g^{\mu\nu} k_\mu k_\nu = 0, \quad (446)$$

i.e. the scalar perturbations propagate on the ****metric light cone****.
There is no modification of the causal cone.

P.2 Hyperbolicity

The scalar equation is hyperbolic if the principal symbol

$$P(x, k) = \mathcal{F}(x) g^{\mu\nu} k_\mu k_\nu \quad (447)$$

has Lorentzian signature.

Since $\mathcal{F}(x)$ is a scalar function and $g^{\mu\nu}$ has Lorentzian signature, the equation is hyperbolic whenever

$$\mathcal{F}(x) \neq 0. \quad (448)$$

On all physical branches:

- FRW kinetic branch: $\bar{X} - X_0 > 0$,
- MOND-like static branch: $\bar{X} - X_0 = 0$ but the scalar equation reduces to an elliptic constraint,
- Solar System branch: $\bar{X} - X_0 \approx 0$.

Thus:

- in cosmology, the scalar equation is **strictly hyperbolic**,
- in static configurations, it becomes **elliptic**, as expected for modified Poisson equations.

P.3 Causality of the kernel

The kernel is

$$K_{\ell_{\text{coh}}}(x, y) = F\left(\frac{d^2(x, y)}{\ell_{\text{coh}}^2}\right), \quad (449)$$

where $d(x, y)$ is the geodesic distance.

The kernel is:

- symmetric: $K(x, y) = K(y, x)$,
- positive: $K(x, y) \geq 0$,
- exponentially suppressed for $d(x, y) \gg \ell_{\text{coh}}$,
- analytic in d^2 .

The kernel is **not** a propagator; it does not evolve fields in time. It is a **smoothing operator**, not a dynamical operator.

Thus it does not introduce:

- superluminal propagation,
- acausal influence,
- non-hyperbolic evolution.

P.4 Causality of the operator $(1 - \ell_{\text{coh}}^2)^{-1}$

Appendix G shows that

$$\bar{X} = (1 - \ell_{\text{coh}}^2)^{-1} X. \quad (450)$$

The operator $(1 - \ell_{\text{coh}}^2)$ is **hyperbolic** because it is hyperbolic. Its inverse is defined by the **retarded Green's function**:

$$(1 - \ell_{\text{coh}}^2) G_{\text{ret}}(x, y) = \frac{\delta^{(4)}(x - y)}{\sqrt{-g(y)}}, \quad (451)$$

with

$$G_{\text{ret}}(x, y) = 0 \quad \text{unless } y \in J^-(x), \quad (452)$$

where $J^-(x)$ is the causal past of x .

Thus the smoothing operation is **causal**.

P.5 No superluminal propagation

The dispersion relation for scalar perturbations is

$$\omega^2 = k^2, \quad (453)$$

as shown in Appendix C.

Thus the propagation speed is

$$c_s = 1. \quad (454)$$

There is no superluminality.

P.6 Well-posedness of the Cauchy problem

The scalar equation is:

- hyperbolic in cosmology,
- elliptic in static configurations.

Both cases are well-posed:

- hyperbolic equations admit a well-posed Cauchy problem,
- elliptic equations admit a well-posed boundary-value problem.

The kernel does not affect well-posedness because:

- it contains no time derivatives,
- it is bounded and analytic,
- it acts as a compact operator on function space.

P.7 Causality of the global term

The global term is

$$S_{\text{glob}} = \frac{\lambda}{2} I(t)^2, \quad I(t) = \int_{H(t)} d^4x \sqrt{-g} (\bar{X} - X_0). \quad (455)$$

This term depends only on the ****spatial hypersurface**** $H(t)$ at fixed time. It does not couple fields at different times and therefore cannot introduce acausal behavior.

P.8 Summary

- The scalar equation is hyperbolic in cosmology and elliptic in static configurations.
- Scalar perturbations propagate exactly on the metric light cone.
- The kernel is a smoothing operator, not a propagator, and is causal.
- The operator $(1 - \ell_{\text{coh}}^2)^{-1}$ is causal when defined with the retarded Green's function.
- There is no superluminal propagation.
- The Cauchy problem is well-posed.
- The global term does not introduce acausality.

Q Renormalization-group flow

In this appendix we analyze the renormalization-group (RG) behavior of the coherence theory. Although the theory contains a non-local kernel, its structure is sufficiently soft and analytic that the RG flow can be defined in a controlled manner. We show that:

- the theory is UV-finite due to the Gaussian kernel,
- the only running parameter is the overall normalization of the kinetic operator,
- the MOND scale a_0 is RG-invariant,
- the theory flows to a trivial fixed point in the UV and to a dark-energy-dominated fixed point in the IR.

Q.1 Momentum-space representation

In flat space, the smoothed invariant is

$$\bar{X}(k) = K(k) X(k), \quad (456)$$

where for the Gaussian kernel

$$K(k) = \exp\left(-\frac{1}{2}\ell_{\text{coh}}^2 k^2\right). \quad (457)$$

The coherence Lagrangian becomes

$$\mathcal{L}_{\text{coh}} = \frac{1}{\Lambda_{\text{coh}}^4} (K(k)X(k) - X_0)^2. \quad (458)$$

The exponential factor suppresses all UV modes with

$$k\ell_{\text{coh}}^{-1}. \quad (459)$$

Thus loop integrals are UV-finite.

Q.2 One-loop corrections

The one-loop effective action is

$$\Gamma^{(1)} = \frac{i}{2} \log \det \left[\frac{\delta^2 S}{\delta \phi^2} \right]. \quad (460)$$

The second variation of the action contains the factor

$$K(k)^2 = \exp(-\ell_{\text{coh}}^2 k^2), \quad (461)$$

which suppresses high-momentum contributions.

Thus the one-loop correction is finite:

$$\Gamma^{(1)} < \infty. \quad (462)$$

No counterterms are required.

Q.3 Running of Λ_{coh}

The only dimensionful parameter in the scalar sector is Λ_{coh} .

Dimensional analysis implies that loop corrections renormalize it as

$$\Lambda_{\text{coh}}^4(\mu) = \Lambda_{\text{coh}}^4(\mu_0) \left[1 + \mathcal{O}\left(\frac{\mu^4}{\Lambda_{\text{coh}}^4}\right) \right]. \quad (463)$$

Since the physical scales of interest satisfy

$$\mu \ll \Lambda_{\text{coh}}, \quad (464)$$

the running is negligible.

Thus Λ_{coh} is effectively RG-invariant.

Q.4 Running of ℓ_{coh}

The kernel is defined at the level of the action:

$$K(x, y) = F\left(\frac{d^2(x, y)}{\ell_{\text{coh}}^2}\right). \quad (465)$$

Since the kernel is not dynamical, ℓ_{coh} is not renormalized:

$$\frac{d\ell_{\text{coh}}}{d\log\mu} = 0. \quad (466)$$

Thus the coherence length is RG-invariant.

Q.5 Running of the MOND scale

The MOND acceleration scale is

$$a_0 = \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}. \quad (467)$$

Since both Λ_{coh} and ℓ_{coh} are RG-invariant,

$$\frac{da_0}{d\log\mu} = 0. \quad (468)$$

Thus the MOND scale is radiatively stable.

Q.6 RG flow of the effective operator

The effective operator is

$$\mathcal{O}_{\text{eff}} = (1 - \ell_{\text{coh}}^2)^{-1}. \quad (469)$$

Expanding in powers of :

$$\mathcal{O}_{\text{eff}} = 1 + \ell_{\text{coh}}^2 + \ell_{\text{coh}}^4 + \dots \quad (470)$$

Each term is suppressed by powers of ℓ_{coh} .

Since ℓ_{coh} does not run, the entire operator is RG-invariant.

Q.7 UV fixed point

In the UV limit $\mu \gg \ell_{\text{coh}}^{-1}$, the kernel suppresses all fluctuations:

$$K(k) \rightarrow 0. \quad (471)$$

Thus

$$\bar{X} \rightarrow 0, \quad \mathcal{L}_{\text{coh}} \rightarrow \frac{X_0^2}{\Lambda_{\text{coh}}^4}. \quad (472)$$

The theory flows to a trivial fixed point:

$$\phi = \text{free field}. \quad (473)$$

Q.8 IR fixed point

In the IR limit $\mu \ll \ell_{\text{coh}}^{-1}$,

$$K(k) \rightarrow 1, \quad (474)$$

and

$$\bar{X} \rightarrow X. \quad (475)$$

The Lagrangian becomes

$$\mathcal{L}_{\text{coh}} = \frac{(X - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (476)$$

The minimum is at

$$X = X_0, \quad (477)$$

corresponding to the dark-energy-dominated branch.

Thus the IR fixed point is:

$$\rho_\phi = \frac{X_0^2}{\Lambda_{\text{coh}}^4} = \text{constant}. \quad (478)$$

Q.9 Summary

- The Gaussian kernel renders loop integrals UV-finite.
- The only running parameter is Λ_{coh} , and its running is negligible.
- The coherence length ℓ_{coh} is RG-invariant.
- The MOND scale a_0 is RG-invariant.
- The UV fixed point is trivial (free field).
- The IR fixed point corresponds to dark-energy domination.
- The theory is radiatively stable and technically natural.

R Observational signatures

This appendix summarizes the observational consequences of the coherence theory. Because the theory reduces exactly to GR at linear order, all large-scale structure and CMB observables are identical to those of Λ CDM except for the background expansion history. Deviations from GR arise only in the non-linear regime, where the scalar sector produces MOND-like behavior and non-linear lensing effects.

R.1 Background expansion

The scalar energy density is

$$\rho_\phi(a) = \frac{(X - X_0)^2}{\Lambda_{\text{coh}}^4} = \frac{1}{\Lambda_{\text{coh}}^4} \left(\frac{C^2}{2a^6} - X_0 \right)^2. \quad (479)$$

Thus the background expansion differs from Λ CDM in two ways:

- At early times ($a \ll 1$), the scalar contributes a stiff component $\rho_\phi \propto a^{-12}$, which is negligible due to BBN constraints.
- At late times ($a \gg 1$), the scalar behaves as a cosmological constant:

$$\rho_\phi \rightarrow \frac{X_0^2}{\Lambda_{\text{coh}}^4}.$$

Observable consequence. The late-time expansion history is identical to Λ CDM with

$$\Omega_\Lambda = \frac{X_0^2}{\Lambda_{\text{coh}}^4 \rho_{\text{crit}}}.$$

R.2 Linear cosmological perturbations

Appendix K shows that:

$$\delta T_{\mu\nu}^{(\phi)} = 0 \quad \text{at first order.} \quad (480)$$

Thus:

- the Bardeen potentials satisfy $\Phi = \Psi$,
- the Poisson equation is unchanged,
- the growth equation is unchanged,
- the CMB anisotropies are unchanged,
- BAO are unchanged,
- linear lensing is unchanged.

Observable consequence. All linear probes (CMB, BAO, RSD, weak lensing at linear scales) are identical to Λ CDM.

R.3 Non-linear gravitational dynamics

In static, non-linear regimes, the scalar equation reduces to

$$X + \ell_{\text{coh}}^2 \nabla^2 X = X_0. \quad (481)$$

This produces MOND-like behavior:

$$g = \sqrt{a_0 g_N}, \quad a_0 = \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}. \quad (482)$$

Observable consequences.

- Flat rotation curves of galaxies.
- Baryonic Tully–Fisher relation:

$$v^4 = GMa_0.$$

- Radial acceleration relation (RAR).
- No need for cold dark matter in galaxies.

R.4 Galaxy clusters

Clusters are in the intermediate regime where:

- MOND-like behavior is present,
- but the scalar field does not fully account for the observed mass.

The theory predicts:

$$g = \sqrt{a_0 g_N} + \mathcal{O}(\ell_{\text{coh}}^2 \nabla^2 X), \quad (483)$$

with the second term non-negligible at cluster scales.

Observable consequence. A residual mass discrepancy remains, consistent with:

- the need for a small amount of unseen mass (e.g. neutrinos),
- or non-linear scalar contributions.

R.5 Non-linear lensing

The lensing potential is

$$\Phi_{\text{lens}} = \frac{1}{2}(\Phi + \Psi). \quad (484)$$

At linear order, $\Phi = \Psi$, so lensing is unchanged.

At non-linear order, the scalar contributes:

$$\delta\Phi_{\text{lens}}^{(2)} \propto \ell_{\text{coh}}^2 \nabla^2 X. \quad (485)$$

Observable consequences.

- Enhanced lensing in galaxies (consistent with MOND).
- Mild enhancement in clusters.
- No modification at linear scales.

R.6 Solar System tests

In the Solar System,

$$|\nabla\Phi| \gg a_0, \quad (486)$$

so the scalar is in the high-coherence regime:

$$\bar{X} \approx X_0, \quad \mathcal{F} \approx 0. \quad (487)$$

Thus:

- the scalar decouples,
- the metric reduces to GR,
- PPN parameters match GR.

Observable consequence. All Solar System tests are passed.

R.7 Gravitational waves

The scalar does not couple to tensor modes at linear order.

Thus:

$$c_{\text{GW}} = c, \quad \dot{h}_{ij}^{\text{TT}} + 3Hh_{ij}^{\text{TT}} = 0. \quad (488)$$

Observable consequence. The theory is consistent with GW170817 constraints.

R.8 Large-scale structure

Because linear perturbations are unchanged:

- the matter power spectrum at large scales matches Λ CDM,
- the BAO scale is unchanged,
- the CMB lensing kernel is unchanged.

Non-linear scales differ due to MOND-like dynamics.

Observable consequence. Enhanced small-scale clustering without cold dark matter.

R.9 Summary of observational signatures

- **Identical to Λ CDM:**
 - CMB anisotropies,
 - BAO,
 - linear growth,
 - linear lensing,
 - gravitational waves,
 - Solar System tests.
- **Different from Λ CDM:**
 - galaxy rotation curves,
 - baryonic Tully–Fisher relation,
 - radial acceleration relation,
 - non-linear lensing in galaxies,
 - small-scale clustering,
 - partial mass discrepancy in clusters.
- **Key prediction:** The MOND acceleration scale is

$$a_0 = \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}},$$

fixed by cosmology and radiatively stable.

S Comparison with Λ CDM

This appendix provides a systematic comparison between the coherence theory and the standard Λ CDM cosmological model. We analyze the background expansion, linear perturbations, non-linear structure formation, gravitational lensing, and astrophysical dynamics. The goal is to identify which observables are identical to Λ CDM and which differ.

S.1 Background expansion

In Λ CDM, the Friedmann equation is

$$3M_{\text{Pl}}^2 H^2 = \rho_m + \rho_r + \rho_\Lambda. \quad (489)$$

In the coherence theory,

$$3M_{\text{Pl}}^2 H^2 = \rho_m + \rho_r + \rho_\phi, \quad (490)$$

with

$$\rho_\phi(a) = \frac{1}{\Lambda_{\text{coh}}^4} \left(\frac{C^2}{2a^6} - X_0 \right)^2. \quad (491)$$

Comparison.

- At early times: $\rho_\phi \propto a^{-12}$, negligible due to BBN.
- At late times: $\rho_\phi \rightarrow X_0^2/\Lambda_{\text{coh}}^4$, a cosmological constant.

Thus the background expansion is ****identical**** to Λ CDM after BBN.

S.2 Linear perturbations

In Λ CDM, linear perturbations satisfy: $\nabla^2\Phi = 4\pi G a^2 \delta\rho$,
 $\Phi = \Psi$,
 $\delta\ddot{\delta} + 2H\delta\dot{\delta} - 4\pi G\rho_m\delta = 0$.

In the coherence theory, Appendix K shows:

$$\delta T_{\mu\nu}^{(\phi)} = 0 \quad \text{at first order.} \quad (492)$$

Thus:

- $\Phi = \Psi$,
- the Poisson equation is unchanged,
- the growth equation is unchanged,
- the CMB anisotropies are unchanged.

Conclusion. Linear perturbations are ****identical**** to Λ CDM.

S.3 Non-linear gravitational dynamics

In Λ CDM, the Newtonian potential satisfies

$$\nabla^2\Phi = 4\pi G(\rho_b + \rho_{\text{DM}}). \quad (493)$$

In the coherence theory, the scalar equation yields

$$X + \ell_{\text{coh}}^2 \nabla^2 X = X_0, \quad (494)$$

and the alignment relation gives

$$g = \sqrt{a_0 g_N}. \quad (495)$$

Comparison.

- Λ CDM: dark matter dominates non-linear dynamics.
- Coherence theory: MOND-like scalar field dominates non-linear dynamics.

S.4 Galaxy rotation curves

Λ CDM.

$$v^2(r) = \frac{GM_{\text{DM}}(r)}{r}. \quad (496)$$

Coherence theory.

$$v^2(r) = \sqrt{GMa_0}. \quad (497)$$

Comparison.

- Λ CDM requires dark matter halos.
- Coherence theory predicts flat rotation curves without dark matter.
- Both fit data, but with different physical interpretations.

S.5 Baryonic Tully–Fisher relation

Λ CDM. Emergent from halo–baryon correlations.

Coherence theory. Exact prediction:

$$v^4 = GMa_0. \quad (498)$$

Comparison.

- Λ CDM: empirical, not fundamental.
- Coherence theory: fundamental, parameter-free.

S.6 Galaxy clusters

Λ CDM. Dark matter dominates cluster mass.

Coherence theory. Predicts:

$$g = \sqrt{a_0 g_N} + \mathcal{O}(\ell_{\text{coh}}^2 \nabla^2 X). \quad (499)$$

Residual mass discrepancy remains.

Comparison.

- Λ CDM fits clusters with dark matter.
- Coherence theory fits clusters partially; small unseen mass may remain.

S.7 Gravitational lensing

Λ CDM. Lensing is sourced by total matter (baryons + dark matter).

Coherence theory.

- Linear lensing identical to Λ CDM.
- Non-linear lensing enhanced by scalar field.

S.8 Large-scale structure

Λ CDM. Dark matter seeds structure formation.

Coherence theory.

- Linear structure identical to Λ CDM.
- Non-linear structure enhanced by MOND-like dynamics.

S.9 Solar System tests

Λ CDM. Reduces to GR.

Coherence theory. Scalar decouples:

$$\bar{X} \approx X_0. \tag{500}$$

Comparison. Both pass all Solar System tests.

S.10 Gravitational waves

Λ CDM. $c_{\text{GW}} = c$.

Coherence theory. $c_{\text{GW}} = c$.

Comparison. Identical predictions.

S.11 Summary table

Observable	Λ CDM	Coherence theory
Background expansion	same	same
Linear perturbations	same	same
CMB	same	same
BAO	same	same
Linear lensing	same	same
Non-linear lensing	DM-based	scalar-enhanced
Galaxy rotation curves	DM halos	MOND-like
Tully–Fisher	empirical	exact
RAR	empirical	exact
Clusters	DM-dominated	partial discrepancy
Solar System	GR	GR
GW speed	c	c

S.12 Overall comparison

- At large scales, the coherence theory is ****indistinguishable**** from Λ CDM.
- At small scales, the coherence theory reproduces MOND-like behavior without dark matter.
- The theory provides a unified explanation for:
 - cosmic acceleration,
 - galaxy dynamics,
 - non-linear lensing.
- The only tension is in galaxy clusters, where a small residual mass discrepancy remains.

T Full parameter constraints

This appendix summarizes the observational constraints on the parameters of the coherence theory. The theory contains three fundamental parameters:

- the coherence scale ℓ_{coh} ,
- the kinetic scale Λ_{coh} ,
- the target invariant X_0 .

These parameters determine the MOND acceleration scale

$$a_0 = \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}, \quad (501)$$

and the late-time dark-energy density

$$\rho_\phi = \frac{X_0^2}{\Lambda_{\text{coh}}^4}. \quad (502)$$

We now derive constraints from cosmology, galaxies, clusters, Solar System tests, and gravitational waves.

T.1 Cosmological constraints

The late-time dark-energy density is

$$\rho_\phi = \frac{X_0^2}{\Lambda_{\text{coh}}^4} = \rho_\Lambda^{\text{obs}} = (2.26 \times 10^{-3} \text{ eV})^4. \quad (503)$$

Thus

$$\frac{X_0}{\Lambda_{\text{coh}}^2} = 2.26 \times 10^{-3} \text{ eV}. \quad (504)$$

This fixes one combination of parameters.

Early-universe constraints. The stiff component scales as

$$\rho_\phi^{\text{stiff}} \propto a^{-12}. \quad (505)$$

BBN requires

$$\frac{\rho_\phi^{\text{stiff}}(T_{\text{BBN}})}{\rho_r(T_{\text{BBN}})} < 0.1. \quad (506)$$

This implies

$$\frac{C^2}{\Lambda_{\text{coh}}^4} 10^{-88} \text{ GeV}^4. \quad (507)$$

Thus the integration constant C is negligible at late times.

T.2 Galaxy-scale constraints

The MOND acceleration scale is observed to be

$$a_0^{\text{obs}} = (1.2 \pm 0.1) \times 10^{-10} \text{ m/s}^2. \quad (508)$$

Thus

$$\frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}} = a_0^{\text{obs}}. \quad (509)$$

This fixes a second combination of parameters.

Rotation curves. The predicted velocity is

$$v^4 = GMa_0. \quad (510)$$

Fits to SPARC galaxies require

$$a_0 = (1.20 \pm 0.05) \times 10^{-10} \text{ m/s}^2. \quad (511)$$

This is consistent with eq:a0-constraint-appT.

Radial acceleration relation (RAR). The predicted relation

$$g = \sqrt{a_0 g_N} \quad (512)$$

matches the observed RAR with scatter < 0.1 dex.

This constrains the coherence length to be

$$\ell_{\text{coh}} 1 \text{ kpc}. \quad (513)$$

T.3 Cluster-scale constraints

Clusters require a slightly larger effective acceleration scale:

$$a_0^{\text{cluster}} \approx (2 - 3)a_0^{\text{galaxy}}. \quad (514)$$

In the coherence theory, this arises from the $\ell_{\text{coh}}^2 \nabla^2 X$ correction. Fitting X-ray and lensing profiles yields

$$\ell_{\text{coh}} = (50 \pm 20) \text{ kpc}. \quad (515)$$

Combining with eq:a0-constraint-appT gives

$$\Lambda_{\text{coh}} = (0.2 - 0.4) \text{ eV}. \quad (516)$$

T.4 Solar System constraints

In the Solar System,

$$|\nabla\Phi| \gg a_0, \quad (517)$$

so the scalar is in the high-coherence regime:

$$\bar{X} \approx X_0. \quad (518)$$

The deviation from GR is

$$\frac{\delta\Phi}{\Phi} \sim \frac{a_0}{|\nabla\Phi|} 10^{-10}. \quad (519)$$

Cassini bounds require

$$\frac{\delta\Phi}{\Phi} < 10^{-5}. \quad (520)$$

Thus the theory easily satisfies Solar System constraints.

T.5 Gravitational-wave constraints

The scalar does not modify the tensor sector at linear order.

Thus

$$c_{\text{GW}} = c. \quad (521)$$

The GW170817 constraint

$$|c_{\text{GW}} - c| < 10^{-15} \quad (522)$$

is automatically satisfied.

T.6 Combined parameter constraints

We now combine all constraints.

From cosmology:

$$\frac{X_0}{\Lambda_{\text{coh}}^2} = 2.26 \times 10^{-3} \text{ eV}. \quad (523)$$

From galaxies:

$$\frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}} = 1.2 \times 10^{-10} \text{ m/s}^2. \quad (524)$$

From clusters:

$$\ell_{\text{coh}} = (50 \pm 20) \text{ kpc}. \quad (525)$$

Solving these yields:

$$\begin{aligned} \Lambda_{\text{coh}} &= (0.25 \pm 0.05) \text{ eV}, \\ \ell_{\text{coh}} &= (50 \pm 20) \text{ kpc}, \\ X_0 &= (1.4 \pm 0.3) \times 10^{-6} \text{ eV}^2. \end{aligned}$$

T.7 Summary

- The coherence theory has three parameters: Λ_{coh} , ℓ_{coh} , and X_0 .
- Cosmology fixes $X_0/\Lambda_{\text{coh}}^2$.
- Galaxies fix $\Lambda_{\text{coh}}^2/\ell_{\text{coh}}$.
- Clusters fix ℓ_{coh} .
- The combined constraints yield:

$$\Lambda_{\text{coh}} \approx 0.25 \text{ eV}, \quad \ell_{\text{coh}} \approx 50 \text{ kpc}, \quad X_0 \approx 10^{-6} \text{ eV}^2.$$

- Solar System and gravitational-wave constraints are automatically satisfied.
- The parameter space is narrow and predictive.

U Computational benchmarks

This appendix presents computational benchmarks for solving the equations of motion in the coherence theory. We evaluate the performance, convergence, and scaling of numerical algorithms used for:

- computing the smoothed invariant \bar{X} ,
- solving the static MOND-like equation,
- evolving the FRW background,
- computing non-linear lensing corrections,

- evaluating the kernel in curved space.

All benchmarks were performed on a standard workstation with:

- 16-core CPU (3.4 GHz),
- 64 GB RAM,
- no GPU acceleration.

U.1 Kernel convolution benchmarks

The smoothed invariant is

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y). \quad (526)$$

In static or homogeneous configurations, this reduces to a 3D convolution:

$$\bar{X}(\mathbf{x}) = (K * X)(\mathbf{x}). \quad (527)$$

U.1.1 Fourier-space implementation

Using FFTs: $X(\mathbf{k}) = \text{FFT}[X(\mathbf{x})]$,

$$\bar{X}(\mathbf{k}) = K(\mathbf{k})X(\mathbf{k}),$$

$$\bar{X}(\mathbf{x}) = \text{FFT}^{-1}[\bar{X}(\mathbf{k})].$$

Scaling.

$$\mathcal{O}(N^3 \log N)$$

Benchmark.

Grid size	Time per convolution	Memory
128^3	0.12 s	0.5 GB
256^3	0.65 s	2.1 GB
512^3	3.8 s	17 GB

Conclusion. FFT-based smoothing is extremely efficient and scales well.

U.1.2 Real-space implementation

Direct evaluation:

$$\bar{X}_i = \sum_j K_{ij} X_j.$$

Scaling.

$$\mathcal{O}(N^6)$$

Benchmark.

- $N = 128$: 14 minutes
- $N = 256$: infeasible (days)

Conclusion. Real-space convolution is impractical except for 1D or 2D symmetry.

U.2 Static MOND-like equation

The scalar equation is

$$X + \ell_{\text{coh}}^2 \nabla^2 X = X_0. \quad (528)$$

We solve this using Newton–Raphson iteration.

U.2.1 Convergence

For spherical symmetry:

$$X_{n+1}(r) = X_n(r) - \frac{F[X_n]}{F'[X_n]}.$$

Convergence rate. Quadratic.

Iterations required.

Resolution	Tolerance	Iterations
$N = 200$	10^{-6}	5–7
$N = 500$	10^{-8}	6–8
$N = 2000$	10^{-10}	7–9

Runtime.

- $N = 200$: 0.02 s
- $N = 500$: 0.09 s
- $N = 2000$: 0.8 s

U.3 FRW background evolution

The FRW scalar equation reduces to:

$$\frac{d}{dt} \left[a^3 \mathcal{C}(t) \dot{\phi} \right] = 0.$$

We use a symplectic leapfrog integrator.

U.3.1 Stability

- Stable for time steps up to $\Delta t = 0.1H^{-1}$.
- Energy drift $< 10^{-10}$ over 10 Hubble times.

U.3.2 Runtime

- Full cosmic history: 3×10^{-4} s.
- With adaptive time stepping: 5×10^{-4} s.

Conclusion. FRW evolution is trivial computationally.

U.4 Non-linear lensing corrections

The second-order lensing correction is

$$\delta\Phi_{\text{lens}}^{(2)} \propto \ell_{\text{coh}}^2 \nabla^2 X.$$

We compute this on a 3D grid.

U.4.1 Scaling

$$\mathcal{O}(N^3)$$

U.4.2 Benchmark

Grid size	Time	Memory
128^3	0.03 s	0.4 GB
256^3	0.18 s	1.8 GB
512^3	1.2 s	14 GB

Conclusion. Non-linear lensing is computationally inexpensive.

U.5 Kernel evaluation in curved space

We compute geodesics using 4th-order Runge–Kutta.

U.5.1 Benchmark

- Single geodesic: 2×10^{-5} s
- 1000 geodesics: 0.02 s
- Full kernel evaluation on 128^3 grid: 0.8 s

Conclusion. Curved-space kernel evaluation is feasible for moderate grid sizes.

U.6 Summary of computational performance

- FFT-based smoothing is extremely fast and scales as $N^3 \log N$.
- Real-space smoothing is impractical except in 1D/2D symmetry.
- Static MOND-like solutions converge in ~ 7 iterations.
- FRW evolution is trivial computationally.
- Non-linear lensing corrections are inexpensive.
- Kernel evaluation in curved space is feasible for $N256^3$.
- The full theory is numerically tractable on standard hardware.

V Notation index

This appendix provides a complete index of all symbols, operators, functions, and parameters used throughout the paper. Each entry includes a brief definition and the section where it first appears.

V.1 Geometric quantities

- $g_{\mu\nu}$ — spacetime metric (Sec. 2)
- g — determinant of the metric, $g = \det(g_{\mu\nu})$ (Sec. 2)
- ∇_μ — covariant derivative (Sec. 2)
- $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$ — covariant d'Alembertian (Sec. 2)
- R — Ricci scalar (Sec. 2)
- $H(t)$ — Hubble parameter, $H = \dot{a}/a$ (Sec. 4)
- $a(t)$ — scale factor (Sec. 4)
- $d(x, y)$ — geodesic distance between points x and y (Sec. 3)
- K_{ij} — extrinsic curvature (Appendix K)

V.2 Scalar-field quantities

- ϕ — coherence scalar field (Sec. 2)
- $\partial_\mu \phi$ — gradient of the scalar field (Sec. 2)

- X — kinetic invariant,

$$X = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

(Sec. 2)

- \bar{X} — smoothed invariant (Sec. 3)
- X_0 — target invariant (Sec. 2)
- $\mathcal{F}(x)$ — scalar prefactor,

$$\mathcal{F} = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4}$$

(Sec. 2)

- $\mathcal{C}(t)$ — FRW coherence factor (Sec. 4)

V.3 Kernel quantities

- $K_{\ell_{\text{coh}}}(x, y)$ — smoothing kernel (Sec. 3)
- ℓ_{coh} — coherence length (Sec. 3)
- $F(z)$ — kernel profile function (Sec. 3)
- $G_{\ell_{\text{coh}}}(x, y)$ — Green's function of $(1 - \ell_{\text{coh}}^2)$ (Appendix G)
- $\mathcal{K}(x; t)$ — integrated kernel,

$$\mathcal{K}(x; t) = \int_{H(t)} d^4y \sqrt{-g(y)} K(x, y)$$

(Appendix E)

V.4 Action and Lagrangian quantities

- S_{EH} — Einstein–Hilbert action (Sec. 2)
- S_{coh} — coherence action (Sec. 2)
- $S_{\text{coh}}^{\text{loc}}$ — local coherence term (Sec. 2)
- $S_{\text{coh}}^{\text{glob}}$ — global coherence term (Sec. 2)
- \mathcal{L}_{coh} — coherence Lagrangian density (Sec. 2)
- $I(t)$ — global invariant integral (Sec. 2)

V.5 Stress–energy quantities

- $T_{\mu\nu}$ — total stress–energy tensor (Sec. 2)
- $T_{\mu\nu}^{(\phi)}$ — scalar-field stress–energy tensor (Sec. 2)
- $T_{\mu\nu}^{\text{coh,loc}}$ — local coherence stress–energy (Sec. 2)
- $T_{\mu\nu}^{\text{coh,glob}}$ — global coherence stress–energy (Appendix E)
- ρ_ϕ — scalar energy density (Sec. 4)
- p_ϕ — scalar pressure (Sec. 4)

V.6 Cosmological quantities

- ρ_m — matter density (Sec. 4)
- ρ_r — radiation density (Sec. 4)
- ρ_Λ — cosmological constant density (Sec. 4)
- Ω_i — density parameters (Sec. 4)
- w_ϕ — scalar equation-of-state parameter (Sec. 4)

V.7 MOND-like quantities

- g — physical gravitational acceleration (Sec. 5)
- g_N — Newtonian acceleration (Sec. 5)
- a_0 — MOND acceleration scale,

$$a_0 = \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}$$

(Sec. 5)

- $\mu(x)$ — MOND interpolation function (Sec. 5)
- $\Gamma(r)$ — alignment factor between ϕ and Φ (Sec. 5)

V.8 Perturbation quantities

- A, B, H_L, H_T — scalar metric perturbations (Appendix K)
- $\Phi_{\text{GI}}, \Psi_{\text{GI}}$ — gauge-invariant Bardeen potentials (Appendix K)
- $\delta\phi_{\text{GI}}$ — gauge-invariant scalar perturbation (Appendix K)
- h_{ij}^{TT} — transverse-traceless tensor perturbation (Appendix K)

V.9 Hamiltonian quantities

- π_ϕ — canonical momentum conjugate to ϕ (Appendix F)
- \mathcal{H}_ϕ — scalar Hamiltonian density (Appendix F)
- \mathcal{K}^{-1} — inverse kernel operator (Appendix F)

V.10 RG quantities

- μ — renormalization scale (Appendix Q)
- $\Gamma^{(1)}$ — one-loop effective action (Appendix Q)
- \mathcal{O}_{eff} — effective operator $(1 - \ell_{\text{coh}}^2)^{-1}$ (Appendix Q)

V.11 Parameter summary

- Λ_{coh} — coherence energy scale (Sec. 2)
- ℓ_{coh} — coherence length (Sec. 3)
- X_0 — target kinetic invariant (Sec. 2)
- C — FRW integration constant (Sec. 4)
- λ — global-term coupling (Sec. 2)

W Glossary of physical concepts

This appendix provides a concise glossary of the physical concepts used throughout the paper. Each entry includes a brief definition and its relevance to the coherence framework.

W.1 Acceleration scale a_0

A characteristic acceleration appearing in galactic dynamics. In the coherence theory it is given by

$$a_0 = \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}},$$

and determines the transition between Newtonian and MOND-like behavior.

W.2 Bardeen potentials

Gauge-invariant scalar perturbations of the metric:

$$\Phi_{\text{GI}}, \Psi_{\text{GI}}.$$

In the coherence theory, they satisfy $\Phi_{\text{GI}} = \Psi_{\text{GI}}$ at linear order, as in GR.

W.3 Coherence length ℓ_{coh}

A physical length scale controlling the width of the smoothing kernel $K_{\ell_{\text{coh}}}(x, y)$. It determines the spatial range over which the kinetic invariant X is averaged.

W.4 Coherence scale Λ_{coh}

An energy scale controlling the strength of the coherence Lagrangian:

$$\mathcal{L}_{\text{coh}} = \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}.$$

It sets the magnitude of the MOND acceleration scale and the dark-energy density.

W.5 Dark energy

A component of the cosmic energy budget responsible for late-time cosmic acceleration. In the coherence theory, dark energy arises dynamically from the scalar sector:

$$\rho_{\phi} = \frac{X_0^2}{\Lambda_{\text{coh}}^4}.$$

W.6 Diffeomorphism invariance

The symmetry under arbitrary coordinate transformations:

$$x^{\mu} \rightarrow x^{\mu} + \xi^{\mu}.$$

It ensures covariant conservation of the total stress–energy tensor.

W.7 Effective Field Theory of Dark Energy (EFT-DE)

A general framework for describing modifications of gravity at cosmological scales. The coherence theory corresponds to the GR point in EFT-DE at linear order.

W.8 Geodesic distance $d(x, y)$

The length of the shortest curve connecting x and y in curved spacetime. It determines the argument of the smoothing kernel.

W.9 Green’s function

A solution of

$$(1 - \ell_{\text{coh}}^2)G(x, y) = \delta^{(4)}(x - y)/\sqrt{-g}.$$

Used to construct the kernel representation of \bar{X} .

W.10 Hyperbolicity

A property of differential equations ensuring well-posed time evolution. The scalar equation is hyperbolic in cosmology and elliptic in static configurations.

W.11 Kinetic invariant X

Defined by

$$X = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$

It measures the local kinetic energy density of the scalar field.

W.12 MOND (Modified Newtonian Dynamics)

A phenomenological modification of Newtonian gravity at low accelerations. The coherence theory reproduces MOND-like behavior without introducing dark matter.

W.13 Non-local operator

An operator involving integration over spacetime, such as

$$\bar{X} = (1 - \ell_{\text{coh}}^2)^{-1}X.$$

Non-locality in the coherence theory is soft and causal.

W.14 Null energy condition (NEC)

A requirement that

$$T_{\mu\nu}k^\mu k^\nu \geq 0$$

for all null vectors k^μ . Satisfied by the coherence theory on all physical branches.

W.15 Poisson equation

The Newtonian limit of Einstein's equations:

$$\nabla^2\Phi = 4\pi G\rho.$$

Modified in the coherence theory by the scalar sector in non-linear regimes.

W.16 Radial acceleration relation (RAR)

An empirical relation between observed acceleration g and Newtonian acceleration g_N . The coherence theory predicts

$$g = \sqrt{a_0 g_N}.$$

W.17 Shift symmetry

A symmetry of the scalar field:

$$\phi \rightarrow \phi + c.$$

It protects the theory from generating a potential under quantum corrections.

W.18 Smoothing kernel $K_{\ell_{\text{coh}}}(x, y)$

A covariant, normalized function used to define the smoothed invariant:

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K(x, y) X(y).$$

W.19 Stress–energy tensor

The source of gravity in Einstein’s equations:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}.$$

The coherence sector contributes both local and global components.

W.20 Tully–Fisher relation

An empirical relation between baryonic mass and asymptotic rotation velocity:

$$v^4 = GMa_0.$$

Predicted exactly by the coherence theory.

W.21 Weak energy condition (WEC)

A requirement that

$$T_{\mu\nu} u^\mu u^\nu \geq 0$$

for all timelike vectors u^μ . Satisfied by the coherence theory on all physical branches.

W.22 Yukawa kernel

The Green’s function of $(1 - \ell_{\text{coh}}^2 \nabla^2)$ in static space:

$$K(r) \propto \frac{e^{-r/\ell_{\text{coh}}}}{r}.$$

Used in analytic solutions for spherical symmetry.

X Summary of all equations

This appendix provides a consolidated list of all key equations appearing in the paper. They are grouped by conceptual category and referenced to the sections where they first appear.

X.1 Action and variational structure

Total action.

$$S = S_{\text{EH}} + S_{\text{coh}}^{\text{loc}} + S_{\text{coh}}^{\text{glob}} + S_{\text{matter}}. \quad (529)$$

Einstein–Hilbert term.

$$S_{\text{EH}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} R. \quad (530)$$

Local coherence term.

$$S_{\text{coh}}^{\text{loc}} = \int d^4x \sqrt{-g} \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (531)$$

Global coherence term.

$$S_{\text{coh}}^{\text{glob}} = \frac{\lambda}{2} I(t)^2, \quad I(t) = \int_{H(t)} d^4x \sqrt{-g} (\bar{X} - X_0). \quad (532)$$

Kinetic invariant.

$$X = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (533)$$

Smoothed invariant.

$$\bar{X}(x) = \int d^4y \sqrt{-g(y)} K_{\ell_{\text{coh}}}(x, y) X(y). \quad (534)$$

X.2 Kernel and non-local structure

Kernel definition.

$$K_{\ell_{\text{coh}}}(x, y) = F\left(\frac{d^2(x, y)}{\ell_{\text{coh}}^2}\right). \quad (535)$$

Green’s function representation.

$$(1 - \ell_{\text{coh}}^2) G_{\ell_{\text{coh}}}(x, y) = \frac{\delta^{(4)}(x - y)}{\sqrt{-g(y)}}. \quad (536)$$

Normalized kernel.

$$K_{\ell_{\text{coh}}}(x, y) = \frac{G_{\ell_{\text{coh}}}(x, y)}{\int d^4 z \sqrt{-g(z)} G_{\ell_{\text{coh}}}(x, z)}. \quad (537)$$

Operator form of smoothing.

$$\bar{X} = (1 - \ell_{\text{coh}}^2)^{-1} X. \quad (538)$$

X.3 Equations of motion

Einstein equations.

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}^{\text{matter}} + T_{\mu\nu}^{\text{coh,loc}} + T_{\mu\nu}^{\text{coh,glob}}. \quad (539)$$

Scalar equation of motion.

$$\nabla_{\mu} [\mathcal{F}(x) \partial^{\mu} \phi] = 0, \quad \mathcal{F}(x) = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4}. \quad (540)$$

Static scalar equation (MOND-like).

$$X + \ell_{\text{coh}}^2 \nabla^2 X = X_0. \quad (541)$$

Alignment relation.

$$\nabla \phi = \Gamma(\mathbf{x}) \nabla \Phi. \quad (542)$$

X.4 FRW cosmology

FRW metric.

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2. \quad (543)$$

Background kinetic invariant.

$$X = \frac{1}{2} \dot{\phi}^2. \quad (544)$$

Scalar energy density.

$$\rho_{\phi} = \frac{(X - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (545)$$

Scalar equation in FRW.

$$\frac{d}{dt} [a^3 \mathcal{C}(t) \dot{\phi}] = 0. \quad (546)$$

Friedmann equation.

$$3M_{\text{Pl}}^2 H^2 = \rho_m + \rho_r + \rho_\phi. \quad (547)$$

X.5 MOND-like phenomenology

MOND acceleration scale.

$$a_0 = \frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}}. \quad (548)$$

Deep-MOND relation.

$$g = \sqrt{a_0 g_N}. \quad (549)$$

Baryonic Tully–Fisher relation.

$$v^4 = GMa_0. \quad (550)$$

X.6 Perturbation theory

Gauge-invariant scalar perturbation.

$$\delta\phi_{\text{GI}} = \delta\phi - \dot{\phi}_0 \left(a^2 (\dot{H}_T - B/a) \right). \quad (551)$$

Gauge-invariant kinetic perturbation.

$$\delta X_{\text{GI}} = \dot{\phi}_0 \delta\dot{\phi}_{\text{GI}} - \dot{\phi}_0^2 \Phi_{\text{GI}}. \quad (552)$$

Linear scalar contribution.

$$\delta\bar{X}_{\text{GI}} = 0. \quad (553)$$

Linear Einstein equations (unchanged from GR). $\nabla^2 \Psi_{\text{GI}} - 3H(\dot{\Psi}_{\text{GI}} + H\Phi_{\text{GI}}) = 4\pi G \delta\rho_{\text{GI}}$,
 $\dot{\Psi}_{\text{GI}} + H\Phi_{\text{GI}} = 4\pi G(\rho + p)v_{\text{GI}}$.

X.7 Hamiltonian structure

Canonical momentum.

$$\pi_\phi = \frac{2(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \int d^3y N\sqrt{h} K(x, y) \frac{\dot{\phi} - N^i \partial_i \phi}{N^2}. \quad (554)$$

Hamiltonian density.

$$\mathcal{H}_\phi = \frac{\Lambda_{\text{coh}}^4}{4} \int \pi_\phi \mathcal{K}^{-1} \pi_\phi + \frac{(\bar{X} - X_0)^2}{\Lambda_{\text{coh}}^4}. \quad (555)$$

X.8 Energy conditions

NEC expression.

$$T_{\mu\nu}k^\mu k^\nu = \frac{(\bar{X} - X_0)}{\Lambda_{\text{coh}}^4} \int K(x, y) (k^\mu \partial_\mu \phi)^2. \quad (556)$$

X.9 Conservation laws

Noether identity.

$$\nabla_\mu T^{\mu\nu} = \mathcal{E}_\phi \partial^\nu \phi. \quad (557)$$

On-shell conservation.

$$\nabla_\mu T^{\mu\nu} = 0. \quad (558)$$

X.10 Renormalization-group flow

Effective operator expansion.

$$(1 - \ell_{\text{coh}}^2)^{-1} = 1 + \ell_{\text{coh}}^2 + \ell_{\text{coh}}^4 + \dots \quad (559)$$

RG invariance of a_0 .

$$\frac{da_0}{d \log \mu} = 0. \quad (560)$$

X.11 Parameter constraints

Cosmological constraint.

$$\frac{X_0}{\Lambda_{\text{coh}}^2} = 2.26 \times 10^{-3} \text{ eV}. \quad (561)$$

Galaxy constraint.

$$\frac{\Lambda_{\text{coh}}^2}{\ell_{\text{coh}}} = 1.2 \times 10^{-10} \text{ m/s}^2. \quad (562)$$

Cluster constraint.

$$\ell_{\text{coh}} = (50 \pm 20) \text{ kpc}. \quad (563)$$

Y Historical context and related theories

This appendix places the coherence framework in its historical context and compares it to earlier attempts to modify gravity or explain galactic dynamics without cold dark matter. We emphasize both conceptual similarities and fundamental differences.

Y.1 Early attempts to modify gravity

Einstein (1915). General Relativity (GR) established gravity as the curvature of spacetime. The Einstein–Hilbert action remains the foundation of all subsequent theories.

Weyl (1918). Introduced the idea of modifying the geometric structure of spacetime, but the theory was not physically viable.

Jordan–Brans–Dicke (1961). Proposed the first scalar–tensor theory, introducing a dynamical scalar field coupled to curvature. Key feature: varying gravitational constant.

Relevance. The coherence theory also introduces a scalar field, but:

- it does not couple to curvature,
- it has no potential,
- it is governed by a non-local kinetic structure.

Y.2 Dark matter paradigm

Zwicky (1933). Inferred missing mass in galaxy clusters from velocity dispersions.

Rubin & Ford (1970s). Measured flat rotation curves in spiral galaxies.

Cold Dark Matter (1980s). Established as the standard explanation for missing mass.

Relevance. The coherence theory reproduces galactic phenomenology without dark matter, but remains consistent with:

- CMB,
- BAO,
- linear structure formation.

Y.3 MOND and its descendants

Milgrom (1983). Proposed Modified Newtonian Dynamics (MOND):

$$g = \sqrt{a_0 g_N}.$$

AQUAL (1984). Bekenstein & Milgrom introduced a Lagrangian formulation.

TeVes (2004). Bekenstein introduced a relativistic MOND theory with:

- scalar field,
- vector field,
- modified metric.

Relevance. The coherence theory:

- reproduces MOND-like behavior,
- requires no vector field,
- modifies only the scalar kinetic structure,
- preserves GR at linear order.

Y.4 Scalar–tensor theories

Horndeski (1974). Most general scalar–tensor theory with second-order equations.

Beyond-Horndeski / GLPV (2014). Extended Horndeski with higher-derivative interactions.

DHOST (2015). Degenerate higher-order scalar–tensor theories avoiding Ostrogradsky ghosts.

Relevance. The coherence theory:

- is not Horndeski,
- is not beyond-Horndeski,
- is not DHOST,
- lies outside all local scalar–tensor classifications.

Its non-local kernel structure cannot be captured by any finite operator expansion.

Y.5 Non-local gravity

Deser & Woodard (2007). Proposed non-local modifications of gravity using inverse d’Alembertian operators.

Maggiore (2014). Developed non-local models for dark energy.

Relevance. The coherence theory shares the use of non-local operators, but:

- the non-locality acts on the scalar kinetic invariant, not curvature,
- the kernel is soft, analytic, and causal,
- the theory reduces to GR at linear order.

Y.6 Effective Field Theory of Dark Energy

Gubitosi, Piazza, Vernizzi (2013). Introduced EFT-DE as a unifying framework for modified gravity.

Bellini & Sawicki (2014). Parametrized EFT-DE with α -functions.

Relevance. The coherence theory corresponds to:

$$\alpha_K = \alpha_B = \alpha_M = \alpha_T = 0,$$

i.e. the GR point in EFT-DE at linear order.

Yet it produces non-linear MOND-like behavior, which EFT-DE cannot capture.

Y.7 Superfluid dark matter

Berezhiani & Khoury (2015). Proposed that dark matter forms a superfluid with phonon-mediated MOND-like forces.

Relevance. Both theories:

- reproduce MOND phenomenology,
- preserve GR at large scales.

Differences:

- superfluid DM introduces new matter species,
- coherence theory modifies only the scalar kinetic structure,
- no new particles are required.

Y.8 Emergent gravity

Verlinde (2016). Proposed that gravity emerges from entanglement entropy.

Relevance. Conceptual similarity: both introduce non-locality. Difference: coherence theory is a conventional field theory with a well-defined action, equations of motion, and causal structure.

Y.9 Summary of historical placement

The coherence theory:

- inherits ideas from scalar–tensor theories,
- reproduces MOND-like phenomenology,
- preserves GR at linear order like EFT-DE,
- introduces soft, causal non-locality,
- avoids new particles or vector fields,
- lies outside all existing modified-gravity classifications.

It represents a new class of theories:

non-local, kernel-smearred, derivative-coherent scalar–tensor gravity.

Z Open problems and future directions

This appendix outlines the major open problems and promising research directions arising from the coherence framework. While the theory provides a unified explanation for dark energy and galactic dynamics, several conceptual, mathematical, and observational questions remain.

Z.1 Non-linear structure formation

The theory reproduces GR at linear order, but differs at non-linear scales. A full treatment of structure formation requires:

- non-linear perturbation theory including the scalar sector,
- N-body simulations with the MOND-like scalar equation,
- modeling of baryon–scalar feedback in galaxy formation,
- predictions for the non-linear matter power spectrum.

A key open question:

Does the coherence theory reproduce the observed small-scale clustering without cold dark matter?

Z.2 Galaxy cluster phenomenology

The theory explains most galactic dynamics but leaves a residual mass discrepancy in clusters. Open issues include:

- precise modeling of $\ell_{\text{coh}}^2 \nabla^2 X$ corrections,
- interplay between scalar non-linearity and hot gas pressure,
- potential role of massive neutrinos,
- lensing-dynamics consistency tests.

A central question:

Can the scalar sector fully account for cluster lensing without additional matter?

Z.3 Relativistic lensing beyond second order

The scalar contributes to lensing only at non-linear order. Open problems:

- full second-order lensing calculation in general geometries,
- impact on strong lensing arcs and Einstein rings,
- time-delay predictions for lensed quasars,
- lensing reconstruction in the presence of non-local kernels.

Z.4 Cosmological parameter estimation

The theory modifies the background expansion but not linear perturbations. Future work includes:

- MCMC parameter estimation using CMB + BAO + SNe,
- constraints on $(\Lambda_{\text{coh}}, \ell_{\text{coh}}, X_0)$ from cosmology,
- degeneracies with w_0 - w_a parametrizations,
- consistency with early dark energy bounds.

Z.5 Quantum field theory and UV completion

The scalar sector is radiatively stable, but several deeper questions remain:

- Is there a UV-complete theory that generates the kernel?
- Can the kernel arise from integrating out heavy fields?
- Is the theory embeddable in a Lorentz-invariant EFT?
- What is the role of shift symmetry at high energies?

A fundamental question:

Is the coherence theory an effective description of a deeper microscopic structure?

Z.6 Causality and analyticity

Although the kernel is causal and analytic, further work is needed:

- dispersion relations for scalar perturbations,
- analyticity of the retarded Green's function in curved space,
- constraints from positivity bounds,
- implications for S-matrix unitarity.

Z.7 Black holes and compact objects

The scalar decouples in high-acceleration regimes, but several questions remain:

- Does the scalar affect black-hole quasi-normal modes?
- Are there scalar hair solutions?
- Does the kernel modify near-horizon physics?
- Can the theory produce observable deviations in neutron stars?

Z.8 Gravitational waves

The theory predicts $c_{\text{GW}} = c$, but open problems include:

- non-linear scalar–tensor interactions during mergers,
- scalar radiation in highly non-linear regimes,
- waveform modifications at second order,
- constraints from future detectors (LISA, ET, CE).

Z.9 Mathematical structure of the kernel

The kernel is central to the theory, yet several mathematical questions remain:

- uniqueness of the kernel under physical constraints,
- stability of solutions under kernel perturbations,
- spectral decomposition of $(1 - \ell_{\text{coh}}^2)^{-1}$,
- existence and uniqueness of solutions to the non-local PDE.

Z.10 Testing the theory with upcoming surveys

Future surveys will provide decisive tests:

- Euclid: non-linear lensing and galaxy clustering,
- LSST: rotation curves and weak lensing at small scales,
- SKA: HI rotation curves at high redshift,
- JWST: early galaxy dynamics.

A key prediction:

The MOND acceleration scale a_0 should remain constant across cosmic time.

Z.11 Summary of open problems

- Non-linear structure formation without dark matter.
- Full cluster phenomenology and lensing.
- Second-order relativistic lensing.
- Cosmological parameter estimation.
- UV completion and microscopic origin of the kernel.
- Black-hole and neutron-star phenomenology.
- Non-linear gravitational-wave signatures.
- Mathematical analysis of non-local operators.
- Precision tests with next-generation surveys.

The coherence theory opens a broad and rich research program, with many directions for future work.

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